机器人学导论

第四章 机器人动力学

Introduction to Robotics

Ch 4. Manipulator Dynamics

陈白帆、谢斌

chenbaifan@csu.edu.cn

中南大学人工智能与机器人实验室

http://faculty.csu.edu.cn/chenbaifan

Man VS Machine ?

Questions

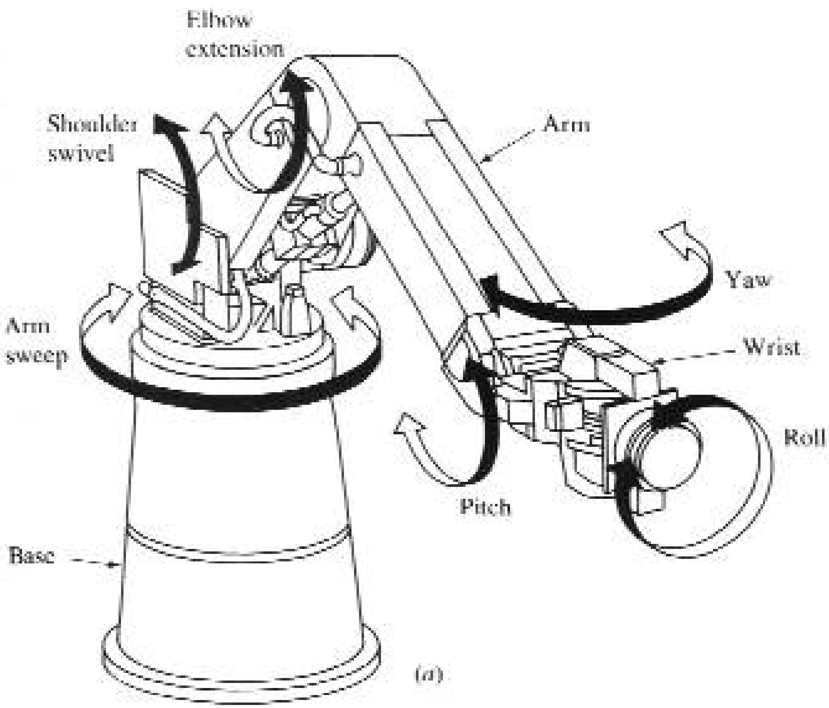
- 1. 如何确定机器人受力与运动之间的关系？
- 2. 给定机器人的运动方式，如何求取对应的驱动力？
- 3. 如何设计一套机器人守门员系统？

Review

- **Introduction to Kinematics of Robotics**
- **Link Description**
- **Frame Attachment**
- **Forward Kinematics**
- **Inverse Kinematics**

Contents

- Introduction



Manipulator Dynamics considers the forces required to cause desired motion.

Considering the equations of motion arises from torques applied by the actuators, or from external forces applied to the manipulator.

- There are two problems related to the dynamics that we wish to solve.

- **Forward Dynamics:** given a torque vector, $\boldsymbol{T}$, calculate the resulting motion of the manipulator, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$. This is useful for simulating the manipulator.

- **Inverse Dynamics:** given a trajectory point, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$, find the required vector of joint torques, $\boldsymbol{T}$. This formulation of dynamics is useful for the problem of controlling the manipulator.

- Two methods for formulating dynamics model:
  - **Newton-Euler dynamic formulation**
    - Newton's equation along with its rotational analog, Euler's equation, describe how forces, inertias, and accelerations relate for rigid bodies, is a "**force balance**" approach to dynamics.
  - **Lagrangian dynamic formulation**
    - Lagrangian formulation is an "**energy-based**" approach to dynamics.

Contents

4.1.1 Kinetic and Potential Energy of a Rigid Body

动能 $$K = \frac{1}{2}M_1\dot{\boldsymbol{x}}_1^2 + \frac{1}{2}M_0\dot{\boldsymbol{x}}_0^2$$

位能 $$P = \frac{1}{2}k(\boldsymbol{x}_1 - \boldsymbol{x}_0)^2 - M_1\boldsymbol{g}\boldsymbol{x}_1 - M_0\boldsymbol{g}\boldsymbol{x}_0$$

耗能 $$D = \frac{1}{2}c(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_0)^2$$

外力功 $$W = \boldsymbol{F}\boldsymbol{x}_1 - \boldsymbol{F}\boldsymbol{x}_0$$

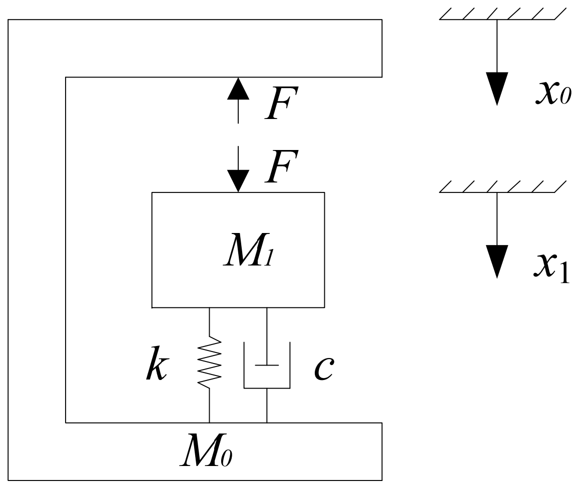
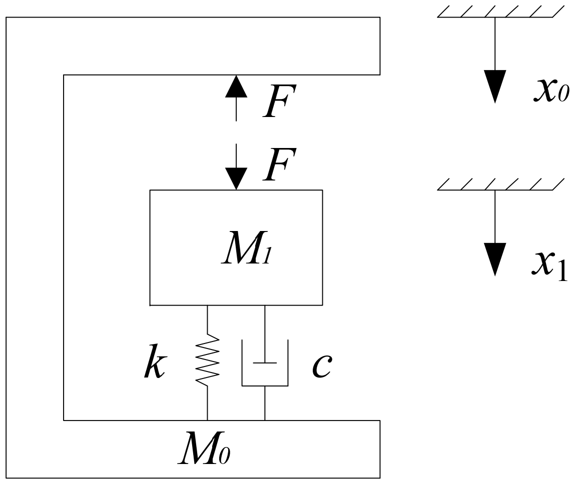


图4.1 一般物体的动能与位能

- $x_0 = 0, x_1$ is a generalized coordinate

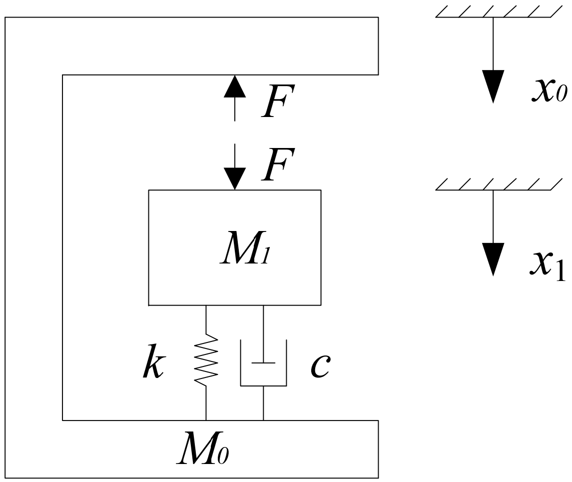
$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_1}\right) - \frac{\partial K}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} + \frac{\partial P}{\partial x_1} = \frac{\partial W}{\partial x_1}$$

① ② ③ ④ ⑤



① Kinetic Energy due to (angular) velocity
② Kinetic Energy due to position (or angle)
③ Dissipation Energy due to (angular) velocity
④ Potential Energy due to position
⑤ External Force or Torque

- x_0 and x_1 are both generalized coordinates



$$M_1\ddot{\boldsymbol{x}}_1 + c(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_0) + k(\boldsymbol{x}_1 - \boldsymbol{x}_0) - M_1\boldsymbol{g} = \boldsymbol{F}$$

$$M_0\ddot{\boldsymbol{x}}_0 + c(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_0) - k(\boldsymbol{x}_1 - \boldsymbol{x}_0) - M_0\boldsymbol{g} = -\boldsymbol{F}$$

Written in Matrices form:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix}\begin{bmatrix} \ddot{\boldsymbol{x}}_1 \\ \ddot{\boldsymbol{x}}_0 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_0 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ -\boldsymbol{F} \end{bmatrix}$$

- Kinetic and Potential Energy of a 2-links manipulator

$$K_1 = \frac{1}{2} m_1 v_1^2, \ v_1 = d_1 \dot{\theta}_1, \ P_1 = m_1 g h_1, \ h_1 = -d_1 \cos\theta_1$$

- Kinetic Energy K_1 and Potential Energy P_1 of link 1

$$K_1 = \frac{1}{2} m_1 d_1^2 \dot{\theta}_1^2, \ P_1 = -m_1 g d_1 \cos\theta_1$$

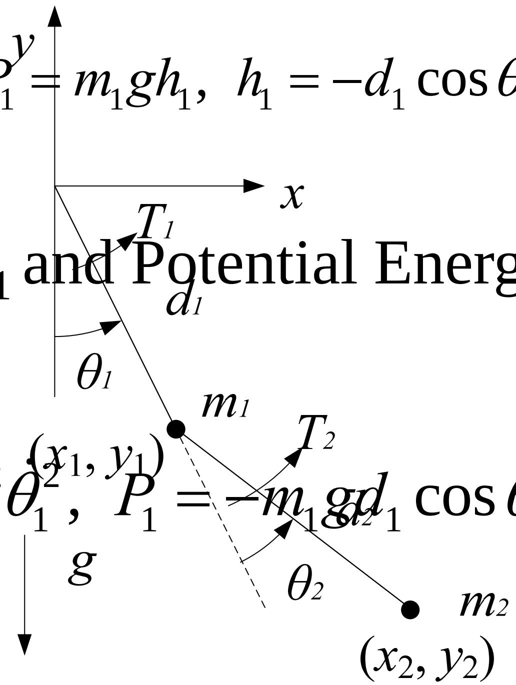
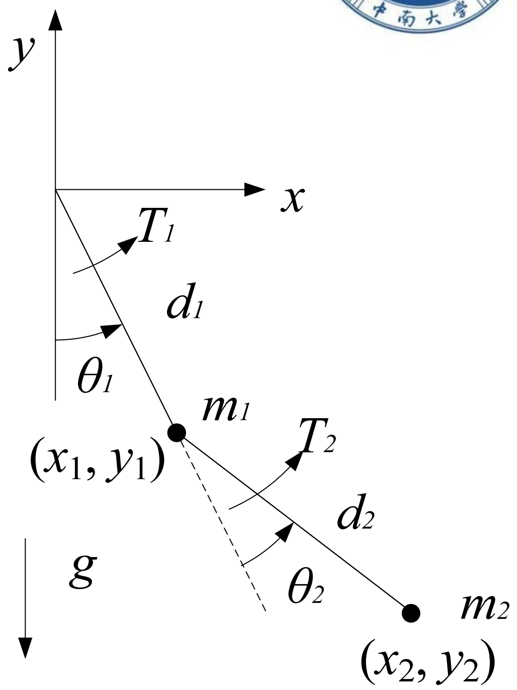


图4.2 二连杆机器手（1）

- Kinetic Energy K_2 and Potential Energy P_2 of link 2

$$K_2 = \frac{1}{2} m_2 v_2^2, \quad P_2 = mgy_2$$



where

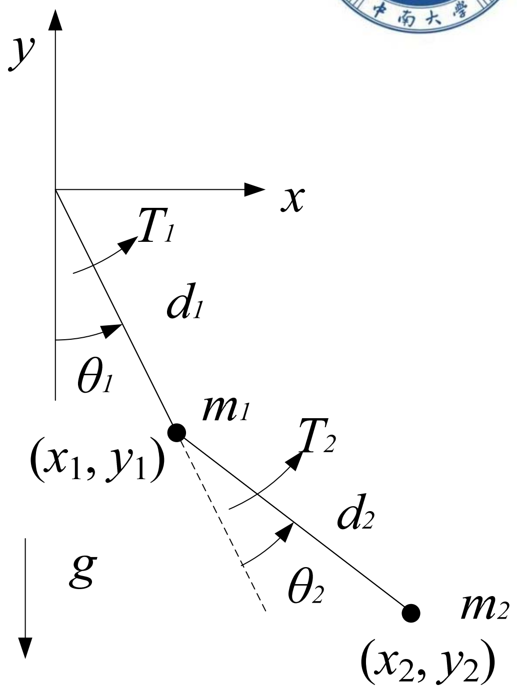
$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$

$$x_2 = d_1 \sin\theta_1 + d_2 \sin(\theta_1 + \theta_2)$$

$$y_2 = -d_1 \cos\theta_1 - d_2 \cos(\theta_1 + \theta_2)$$

$$\Rightarrow \begin{cases} K_2 = \frac{1}{2} m_2 d_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 d_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2 d_1 d_2 \cos\theta_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) \\ P_2 = -m_2 g d_1 \cos\theta_1 - m_2 g d_2 \cos(\theta_1 + \theta_2) \end{cases}$$

- Total Kinetic and Potential Energy of a 2-links manipulator are



$$K = K_1 + K_2$$

$$= \frac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 d_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$+ m_2 d_1 d_2 \cos\theta_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \qquad (4.3)$$

$$P = P_1 + P_2$$

$$= -(m_1 + m_2) g d_1 \cos\theta_1 - m_2 g d_2 \cos(\theta_1 + \theta_2) \qquad (4.4)$$

Contents

4.1.2 Two Solutions for Dynamic Equation

- **Langrangian Function** L is defined as:

$$L = K - P \tag{4.1}$$

Kinetic Energy   Potential Energy

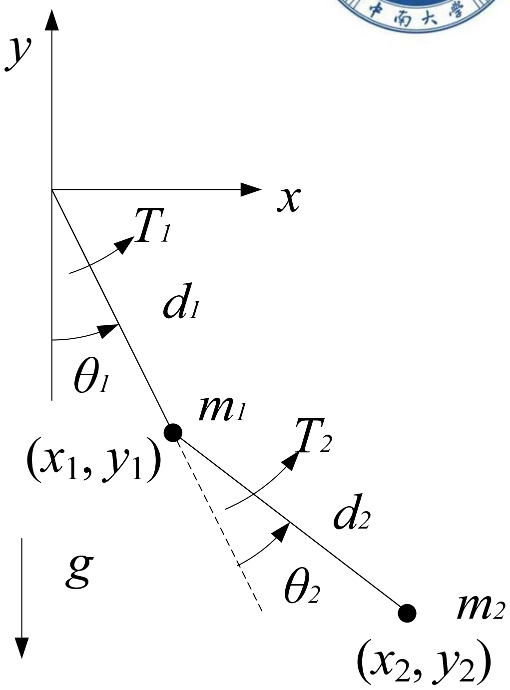
- Dynamic Equation of the system (Langrangian Equation):

$$\boldsymbol{F}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, i = 1,2,\cdots n \tag{4.2}$$

where q_i is the generalized coordinates, $\dot{q}_i$ represent corresponding velocity, F_i stand for corresponding torque or force on the ith coordinate.

Lagrangian Formulation

Lagrangian Function L of a 2-links manipulator:



$$L = K - P$$

$$= \frac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)$$

$$+ m_2 d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) + (m_1 + m_2)g d_1 \cos\theta_1 + m_2 g d_2 \cos(\theta_1 + \theta_2) \quad (4.5)$$

$$\boldsymbol{F}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, i = 1,2,\cdots n$$

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$= [(m_1 + m_2)d_1^2 + m_2 d_2^2 + 2m_2 d_1 d_2 \cos\theta_2]\ddot{\theta}_1 + (m_2 d_2^2 + m_2 d_1 d_2 \cos\theta_2)\ddot{\theta}_2$$

$$- 2m_2 d_1 d_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 d_1 d_2 \sin\theta_2 \dot{\theta}_2^2$$

$$+ (m_1 + m_2) g d_1 \sin\theta_1 + m_2 g d_2 \sin(\theta_1 + \theta_2)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

$$= (m_2 d_2^2 + m_2 d_1 d_2 \cos\theta_2)\ddot{\theta}_1 + m_2 d_2^2 \ddot{\theta}_2 + m_2 d_1 d_2 \sin\theta_2 \dot{\theta}_1^2 + m_2 g d_2 \sin(\theta_1 + \theta_2)$$

$$T_1 = D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2 + D_{111}\dot{\theta}_1^2 + D_{122}\dot{\theta}_2^2 + D_{112}\dot{\theta}_1\dot{\theta}_2 + D_{121}\dot{\theta}_2\dot{\theta}_1 + D_1$$

$$T_2 = D_{21}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 + D_{211}\dot{\theta}_1^2 + D_{222}\dot{\theta}_2^2 + D_{212}\dot{\theta}_1\dot{\theta}_2 + D_{221}\dot{\theta}_2\dot{\theta}_1 + D_2$$

$$D_{11} = (m_1 + m_2)d_1^2 + m_2 d_2^2 + 2m_2 d_1 d_2 \cos\theta_2$$

$$D_{22} = m_2 d_2^2$$

$$D_{12} = D_{21} = m_2 d_2^2 + m_2 d_1 d_2 \cos\theta_2 = m_2(d_2^2 + d_1 d_2 \cos\theta_2)$$

$$D_{111} = 0$$

$$D_{122} = -m_2 d_1 d_2 \sin\theta_2$$

$$D_{211} = m_2 d_1 d_2 \sin\theta_2$$

$$D_{222} = 0$$

$$D_{112} = D_{121} = -m_2 d_1 d_2 \sin\theta_2$$

$$D_{212} = D_{221} = 0$$

$$D_1 = (m_1 + m_2) g d_1 \sin\theta_1 + m_2 g d_2 \sin(\theta_1 + \theta_2)$$

$$D_2 = m_2 g d_2 \sin(\theta_1 + \theta_2)$$

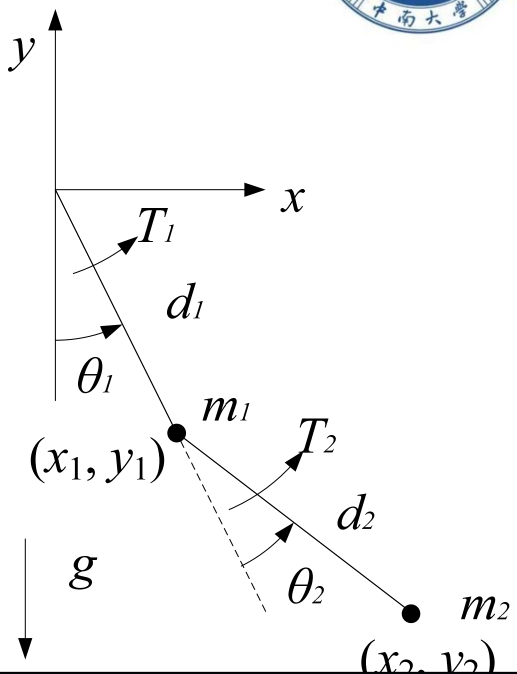
4.1.2 Two Solutions for Dynamic Equation

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



有效惯量(effective inertial)：关节*i*的加速度在关节*i*上产生的惯性力

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \boxed{D_{11}} & D_{12} \\ D_{21} & \boxed{D_{22}} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

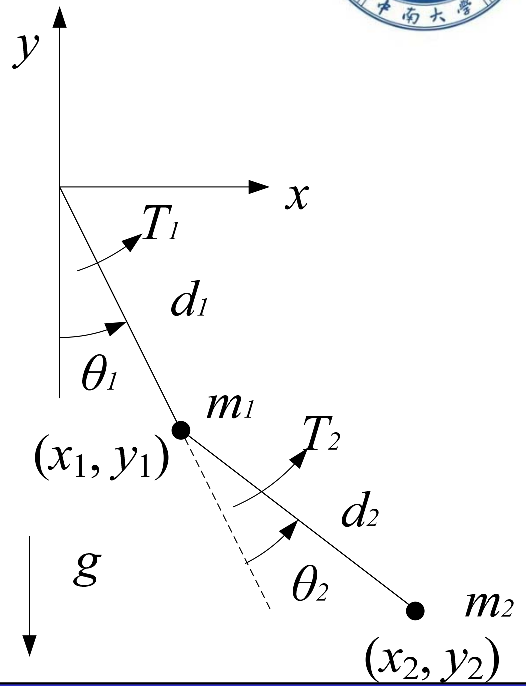
4.1.2 Two Solutions for Dynamic Equation

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



耦合惯量(coupled inertial)：关节*i,j*的加速度在关节*j,i*上产生的惯性力

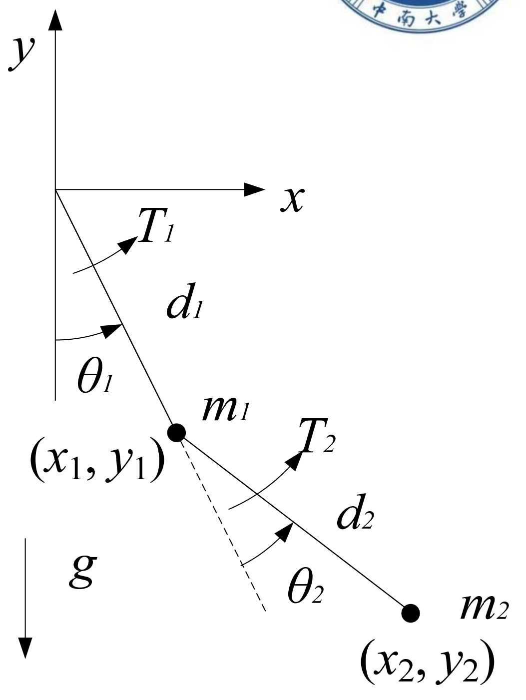
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & \boxed{D_{12}} \\ \boxed{D_{21}} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



向心加速度(acceleration centripetal)系数：关节*i*,*j*的速度在关节*j*,*i*上产生的向心力

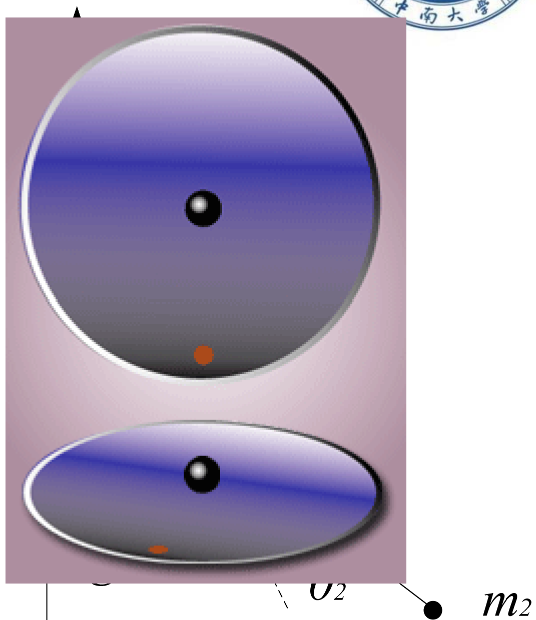
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \boxed{\begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



哥氏加速度(Coriolis accelaration)系数：
关节j,k的速度引起的在关节i上产生的哥氏力(Coriolis force)

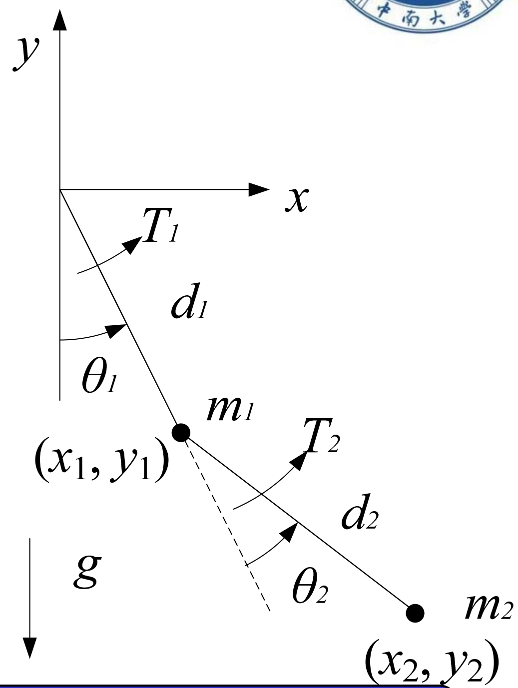
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \boxed{\begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \qquad (4.10)$$

- Lagrangian Formulation

Dynamic Equations:

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \qquad (4.6)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} \qquad (4.7)$$



Written in Matrices Form:

重力项(gravity)：关节*i,j*处的重力

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} + \boxed{\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}} \qquad (4.10)$$

- 假设无重力、静止下，求两种情况的力矩。

（1）关节2锁定（$\ddot{\theta}_2 = 0$）

$$T_1 = D_{11}\ddot{\theta}_1$$

$$T_2 = D_{12}\ddot{\theta}_1$$

（2）关节2不受约束（$T_2 = 0$）

$$T_2 = D_{12}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 = 0 \Rightarrow \ddot{\theta}_2 = -\frac{D_{12}}{D_{22}}\ddot{\theta}_1$$

$$T_1 = (D_{11} - \frac{D_{12}^2}{D_{22}})\ddot{\theta}_1$$

- 取 d_1=d_2=1，m_1=2，计算m_2=1,4和100（分别表示机械手在**地面空载**、**地面满载**和**在外空间负载**的三种不同情况；在外空间由于失重而允许有较大的负载）三个不同数值下各系数的数值。

注意：有效惯量的变化将对机械手的控制产生显著影响！

表4.1

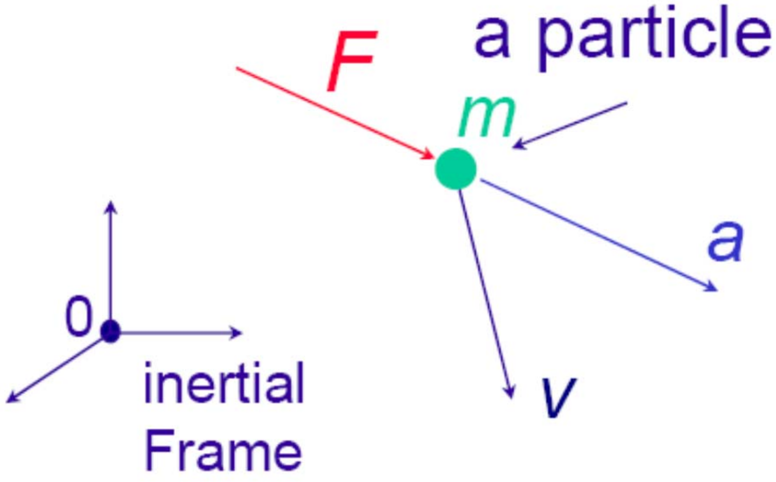
负载	θ_2	$\cos\theta_2$	D_{11}	D_{12}	D_{22}	I_1	I_2
地面空载	0° 90° 180° 270°	1 0 -1 0	6 4 2 4	2 1 0 1	1 1 1 1	6 4 2 4	2 3 2 3
地面满载	0° 90° 180° 270°	1 0 -1 0	18 10 2 10	8 4 0 4	4 4 4 4	18 10 2 10	2 6 2 6
外空间负载	0° 90° 180° 270°	1 0 -1 0	402 202 2 202	200 100 0 100	100 100 100 100	402 202 2 202	2 102 2 102

Contents

4.1.2 Two Solutions for Dynamic Equation

- Newton-Euler Dynamic Formulation

Newton's Law



$$F = m\,a$$

$$\frac{d}{dt}(mv) = F$$

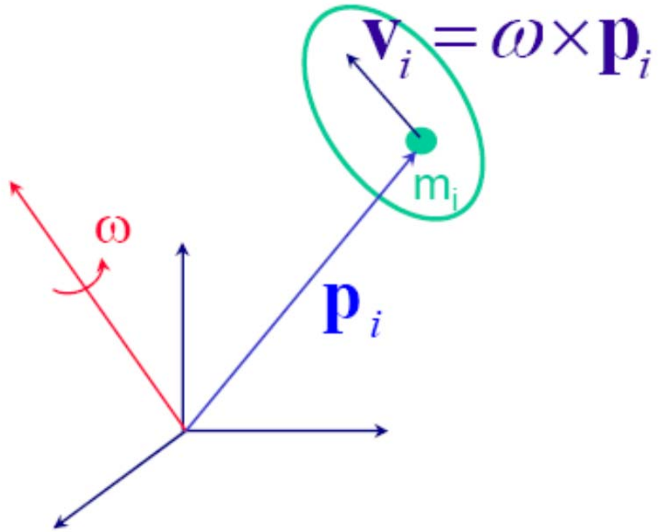
rate of change of the linear momentum is equal to the applied force

Linear Momentum

$$\varphi = mv$$

- Newton-Euler Dynamic Formulation

Rotational Motion



Angular Momentum

$$\sum_i \mathrm{p}_i \times m_i \mathrm{v}_i$$

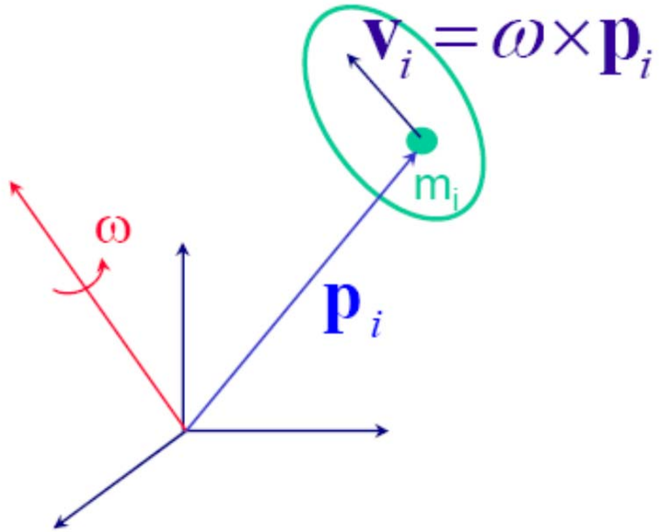
$$\Rightarrow \phi = \sum_i m_i \mathrm{p}_i \times (\omega \times \mathrm{p}_i)$$

$$m_i \to \rho dv \qquad (\rho : \text{density})$$

4.1.2 Two Solutions for Dynamic Equation

- Newton-Euler Dynamic Formulation

Rotational Motion



Angular Momentum

$$\phi = \int_V \mathrm{p} \times (\omega \times \mathrm{p}) \rho dv$$

$$\Rightarrow \phi = \left[\int_V -\hat{\mathrm{p}}\hat{\mathrm{p}} \rho dv \right] \omega$$

Inertia Tensor

$$\phi = I \omega$$

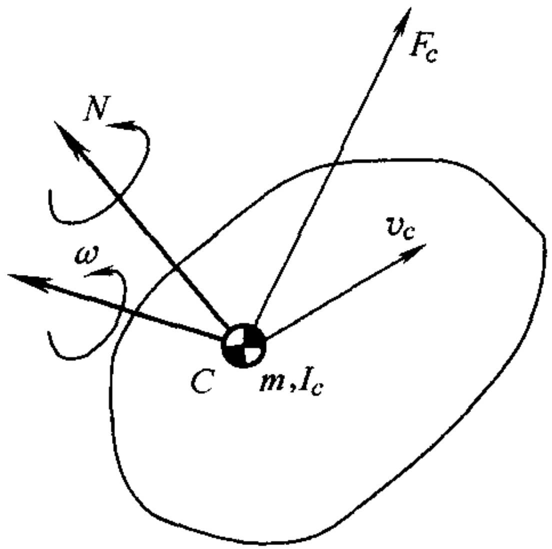
4.1.2 Two Solutions for Dynamic Equation

- Newton-Euler Dynamic Formulation

$$m\dot{v}_c = F_c$$ (**Newton Equation**)

$$I_c\dot{\omega} + \omega \times (I_c\omega) = N$$

(**Euler Equation**)



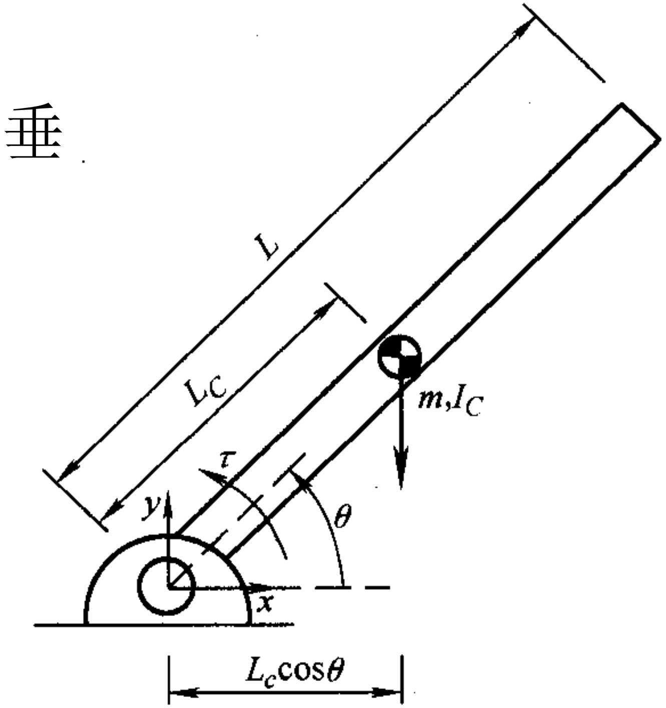
where m is the mass of a rigid body, $I_C \in R^{3\times3}$ represent inertia tensor, F_C is the external force on the center of gravity, N is the torque on the rigid body, v_C represent the translational velocity, while ω is the angular velocity.

- 例1. 求解下图所示的1自由度机械手的运动方程式，在这里，由于关节轴制约连杆的运动，所以可以将运动方程式看作是绕固定轴的运动。

- 解：假设绕关节轴的惯性矩为 I，取垂直纸面的方向为 z 轴，则有

$$I\dot{\omega}=\begin{bmatrix}0\\0\\I\ddot{\theta}\end{bmatrix}\qquad \omega\times I\omega=\begin{bmatrix}0\\0\\\dot{\theta}\end{bmatrix}\times\begin{bmatrix}0\\0\\I\dot{\theta}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

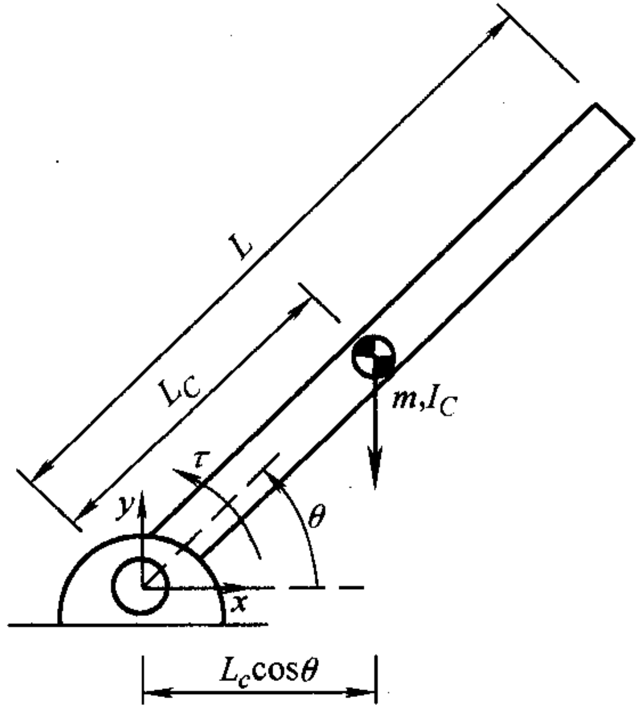
$$N=\begin{bmatrix}0\\0\\\tau-mgL_c\cos\theta\end{bmatrix}$$



1自由度机械手

$$I\dot{\omega}=\begin{bmatrix}0\\0\\I\ddot{\theta}\end{bmatrix}\qquad \omega\times I\omega=\begin{bmatrix}0\\0\\\dot{\theta}\end{bmatrix}\times\begin{bmatrix}0\\0\\I\dot{\theta}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$N=\begin{bmatrix}0\\0\\\tau-mgL_c\cos\theta\end{bmatrix}$$



由欧拉运动方程式 $I_C\dot{\omega}+\omega\times(I_C\omega)=N$

$$I\ddot{\theta}+mgL_c\cos\theta=\tau$$

该式即为1自由度机械手的欧拉运动方程式。

4.1.2 Two Solutions for Dynamic Equation

- **Langrangian Function** L is defined as:

$$L = K - P \quad (4.1)$$

Kinetic Energy Potential Energy

- Dynamic Equation of the system (Langrangian Equation):

$$\boldsymbol{F}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, i = 1,2,\cdots n \quad (4.2)$$

where q_i is the generalized coordinates, $\dot{q}_i$ represent corresponding velocity, F_i stand for corresponding torque or force on the ith coordinate.

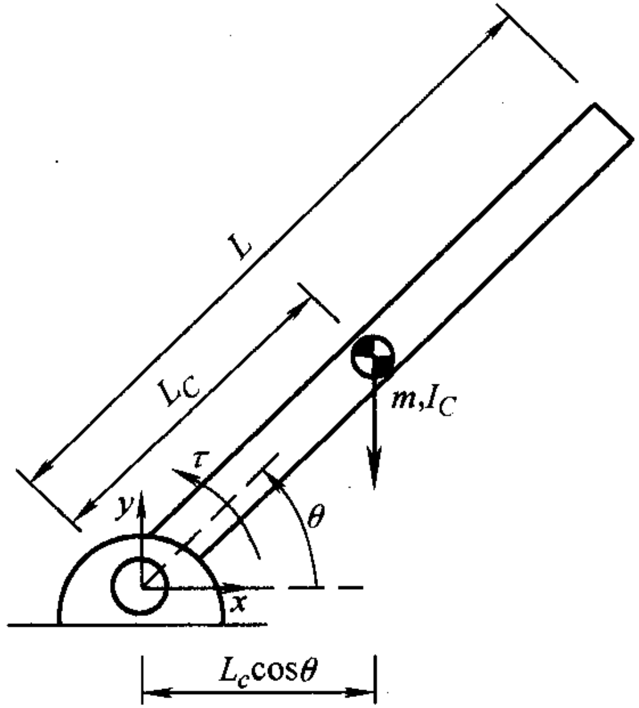
例2.通过拉格朗日运动方程式求解之前推导的1自由度机械手。

解：假设θ为广义坐标，则有

$$K=\frac{1}{2}I\dot{\theta}^2 \qquad P=mgL_c\sin\theta$$

$$L=K-P=\frac{1}{2}I\dot{\theta}^2-mgL_c\sin\theta$$

$$\frac{\partial L}{\partial\dot{\theta}}=I\dot{\theta} \qquad \frac{\partial L}{\partial\theta}=-mgL_c\cos\theta$$



由拉格朗日运动方程 $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\right)-\frac{\partial L}{\partial q}=\tau$

$$\Rightarrow \quad I\ddot{\theta}+mgL_c\cos\theta=\tau$$

4.1.2 Two Solutions for Dynamic Equation

- 我们研究动力学的重要目的之一是为了对机器人的运动进行有效控制，以实现预期的轨迹运动。常用的方法有牛顿—欧拉法、拉格朗日法等。
- 牛顿—欧拉动力学法是利用牛顿力学的刚体力学知识导出逆动力学的递推计算公式，再由它归纳出机器人动力学的数学模型——机器人矩阵形式的运动学方程；
- 拉格朗日法是引入拉格朗日方程直接获得机器人动力学方程的解析公式，并可得到其递推计算方法。

4.1.2 Two Solutions for Dynamic Equation

- 对多自由度的机械手，拉格朗日法可以**直接推导**运动方程式，但随着自由度的增多演算量将**大量增加**。
- 与此相反，牛顿—欧拉法着眼于每一个连杆的运动，即便对于多自由度的机械手其**计算量也不增加**，因此算法易于编程。由于推导出的是一系列公式的组合，要注意**惯性矩阵等的选择和求解**问题。
- 进一步的问题请参考相关文献资料。

Contents

- Forming dynamic equation of any manipulator described by a series of A-matrices:

  (1) Computing the **Velocity** of any given point;

  (2) Computing total **Kinetic Energy**;

  (3) Computing total **Potential Energy**;

  (4) Forming **Lagrangian Function** of the system;

  (5) Forming Dynamic Equation through **Lagrangian Equation**.

4.2.1 Computing the Velocity

- Velocity of point P on link-3:

$$ {}^{0}\boldsymbol{v}_p = \frac{d}{dt}({}^{0}\boldsymbol{r}_p) = \frac{d}{dt}(T_3\,{}^{3}\boldsymbol{r}_p) = \dot{T}_3\,{}^{3}\boldsymbol{r}_p $$

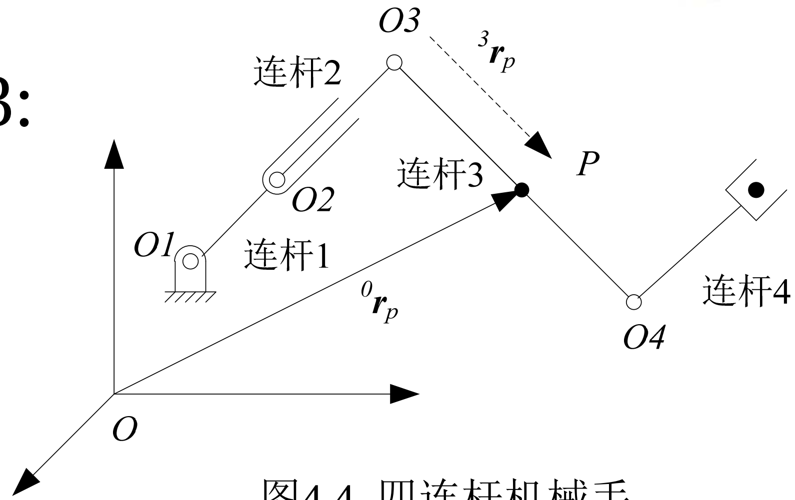


图4.4 四连杆机械手

- Velocity of any given point on link-i:

$$ \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \left(\sum_{j=1}^{i} \frac{\partial T_i}{\partial q_j} \dot{q}_j \right) {}^{i}\boldsymbol{r} \qquad (4.15) $$

- Acceleration of point P:

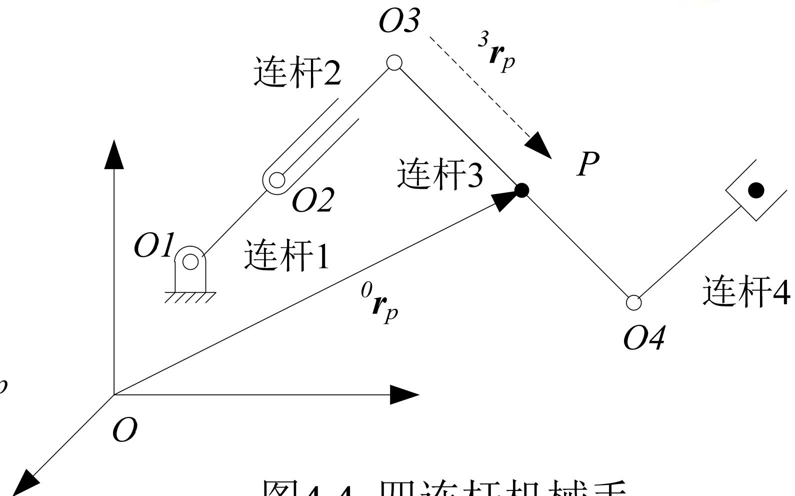


图4.4 四连杆机械手

$$^0\boldsymbol{a}_p = \frac{d}{dt}(^0\boldsymbol{v}_p) = \frac{d}{dt}(\dot{T}_3\,{}^3\boldsymbol{r}_p) = \frac{d}{dt}\left(\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_i}\dot{q}_i\right){}^3\boldsymbol{r}_p$$

$$= \left(\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_i}\frac{d}{dt}\dot{q}_i\right)\left({}^3\boldsymbol{r}_p\right) + \left(\sum_{k=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 T_3}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j\right)\left({}^3\boldsymbol{r}_p\right)$$

$$= \left(\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_i}\ddot{q}_i\right)\left({}^3\boldsymbol{r}_p\right) + \left(\sum_{k=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 T_3}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j\right)\left({}^3\boldsymbol{r}_p\right)$$

4.2.1 Computing the Velocity

- Square of velocity

$$
({}^0\boldsymbol{v}_p)^2 = ({}^0\boldsymbol{v}_p)\cdot({}^0\boldsymbol{v}_p) = Trace[({}^0\boldsymbol{v}_p)\cdot({}^0\boldsymbol{v}_p)^T]
$$

$$
= Trace\left[\sum_{j=1}^{3}\frac{\partial T_3}{\partial q_j}\dot{q}_j({}^3\boldsymbol{r}_p)\cdot\sum_{k=1}^{3}\left(\frac{\partial T_3}{\partial q_k}\dot{q}_k\right)({}^3\boldsymbol{r}_p)^T\right]
$$

$$
= Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial T_3}{\partial q_j}({}^3\boldsymbol{r}_p)({}^3\boldsymbol{r}_p)^T\frac{\partial T_3}{\partial q_k}^T\dot{q}_j\dot{q}_k\right]
$$

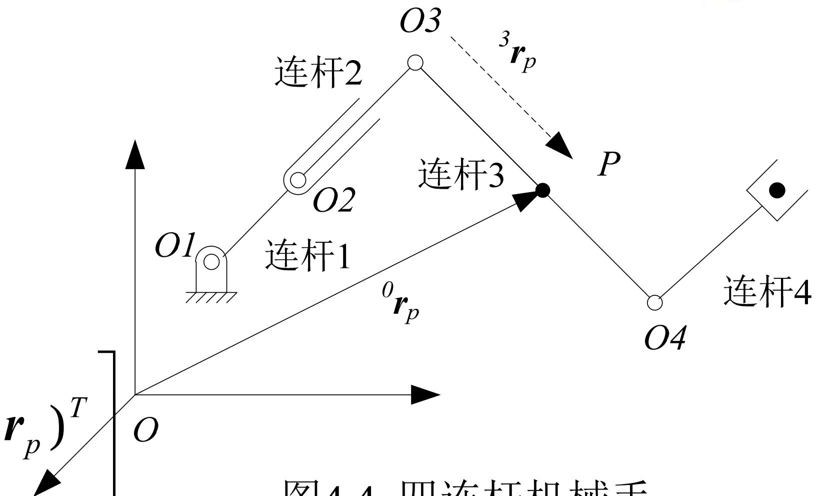


图4.4 四连杆机械手

The trace of an square matrix is defined to be the sum of the diagonal elements.

4.2.1 Computing the Velocity

- Square of velocity of any given point:

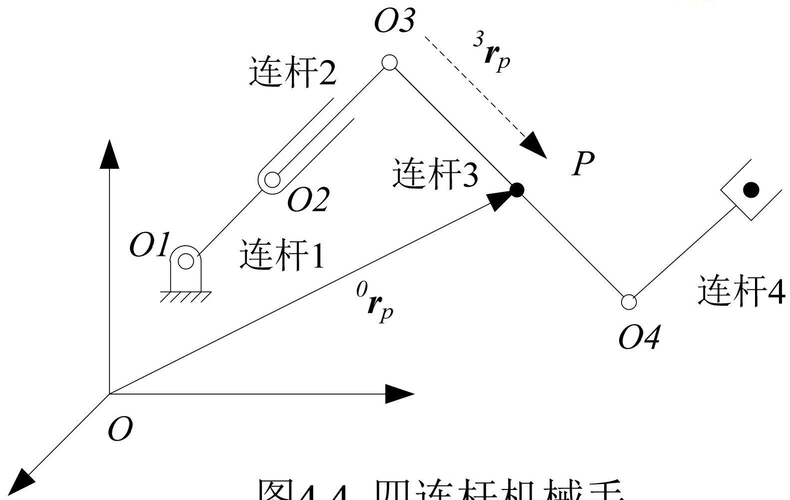


图4.4 四连杆机械手

$$v^2 = \left(\frac{dr}{dt}\right)^2 = Trace\left[\sum_{j=1}^{i}\frac{\partial T_i}{\partial q_j}\dot{q}_j\,{}^i\boldsymbol{r}\sum_{k=1}^{i}\left(\frac{\partial T_i}{\partial q_k}\dot{q}_k\,{}^i\boldsymbol{r}\right)^T\right]$$

$$= Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial T_i}{\partial q_k}\,{}^i\boldsymbol{r}\,{}^i\boldsymbol{r}^T\left(\frac{\partial T_i}{\partial q_k}\right)^T\dot{q}_k\dot{q}_k\right] \qquad (4.16)$$

- Computing the Kinetic Energy

令连杆3上任一质点P的质量为dm，则其动能为：

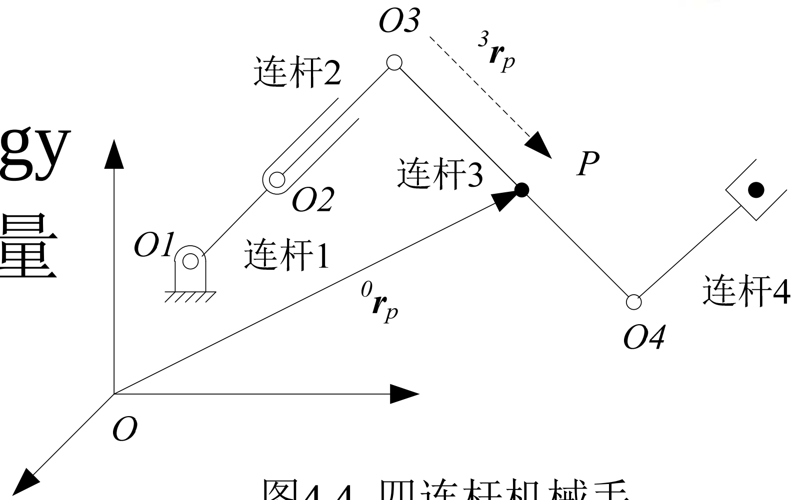


图4.4 四连杆机械手

$$dK_3 = \frac{1}{2} v_p^2 dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial T_3}{\partial q_i} {}^3r_p ({}^3r_p)^T \left(\frac{\partial T_3}{\partial q_k} \right)^T \dot{q}_i \dot{q}_k \right] dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial T_3}{\partial q_i} ({}^3r_p dm {}^3r_p{}^T)^T \left(\frac{\partial T_3}{\partial q_k} \right)^T \dot{q}_i \dot{q}_k \right]$$

- Kinetic Energy of any particle on link-i with position vector ${}^i\boldsymbol{r}$:

$$dK_i = \frac{1}{2} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial T_i}{\partial q_j}{}^j\boldsymbol{r}\,{}^i\boldsymbol{r}^T\frac{\partial T_i^T}{\partial q_k}\;\dot{q}_j\dot{q}_k\right]dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial T_i}{\partial q_j}({}^i\boldsymbol{r}dm\,{}^i\boldsymbol{r}^T)^T\frac{\partial T_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

- Kinetic Energy of link-3:

$$K_3 = \int_{\text{link}3} dK_3 = \frac{1}{2} Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial T_3}{\partial q_j}(\int_{\text{link}3}{}^3\boldsymbol{r}_p\,{}^3\boldsymbol{r}_p{}^T dm)\left(\frac{\partial T_3}{\partial q_k}\right)^T\dot{q}_j\dot{q}_k\right]$$

- Kinetic Energy of any given link-i:

$$K_i = \int_{\text{link } i} dK_i$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} I_i \left(\frac{\partial T_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k \right] \qquad (4.17)$$

- Total Kinetic Energy of the manipulator:

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2} \sum_{i=1}^{n} Trace\left[\sum_{j=1}^{n} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^{T}}{\partial q_k} \dot{q}_i \dot{q}_k \right] \qquad (4.19)$$

- Computing the Potential Energy

Potential Energy of a object (mass *m*) at *h* height:

$$P = mgh$$

so the Potential Energy of any particle on link-i with position vector ${}^i\boldsymbol{r}$:

$$\boldsymbol{g}^T = [g_x, g_y, g_z, 1] \qquad dP_i = -dm\boldsymbol{g}^{T\,0} r = -\boldsymbol{g}^T T_i\, {}^i r dm$$

where

$$P_i = \int_{\text{link } i} dP_i = -\int_{\text{link } i} \boldsymbol{g}^T T_i\, {}^i r dm = -\boldsymbol{g}^T T_i \int_{\text{link } i} {}^i r dm$$

$$= -\boldsymbol{g}^T T_i m_i\, {}^i r_i = -m_i \boldsymbol{g}^T T_i\, {}^i r_i$$

- Potential Energy of any particle on link-i with position vector ${}^{i}\boldsymbol{r}$:

$$dP_i = -dm\boldsymbol{g}^{T\,0}r = -\boldsymbol{g}^T T_i{}^i r dm$$

- Total Potential Energy of the manipulator:

$$P = \sum_{i=1}^{n}(P_i - P_{ai}) \approx \sum_{i=1}^{n} P_i$$
$$= -\sum_{i=1}^{n} m_i \boldsymbol{g}^T T_i{}^i r_i \qquad (4.21)$$

4.2.3 Forming the Dynamic Equation

- Lagrangian Function

$$L = K_t - P$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_i} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \frac{1}{2}\sum_{i=1}^{n} I_{ai}\dot{q}_i^2 + \sum_{i=1}^{n} m_i \boldsymbol{g}^T T_i {}^i r_i ,$$

$$n = 1,2\cdots \qquad (4.22)$$

4.2.3 Forming the Dynamic Equation

- Derivative of Lagrangian function

$$\frac{\partial L}{\partial \dot{q}_p} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^{\ T}}{\partial q_k}\right)\dot{q}_k$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_i} I_i \frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j + I_{ap}\dot{q}_p$$

$$p = 1,2,\cdots n$$

4.2.3 Forming the Dynamic Equation

- According to Eq.(4.18), I_i is a symmetric matrix, lead to

$$Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^T}{\partial q_k}\right) = Trace\left(\frac{\partial T_i}{\partial q_k} I_i^T \frac{\partial T_i^T}{\partial q_j}\right) = Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_j}\right)$$

$$\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=1}^{n} \sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right) \dot{q}_k + I_{ap} \dot{q}_p$$

4.2.3 Forming the Dynamic Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right)\ddot{q}_k + I_{ap}\ddot{q}_p$$

$$+\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_i}\right)\dot{q}_j \dot{q}_k$$

$$+\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_k} I_i \frac{\partial T_i^T}{\partial q_i}\right)\dot{q}_j \dot{q}_k$$

$$= \sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right)\ddot{q}_k + I_{ap}\ddot{q}_p$$

$$+2\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

4.2.3 Forming the Dynamic Equation

$$\frac{\partial L}{\partial q_p} = \frac{1}{2}\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

$$+\frac{1}{2}\sum_{i=p}^{n}\sum_{i=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_k \partial q_p} I_i \frac{\partial T_i^t}{\partial q_j}\right)\dot{q}_j \dot{q}_k + \sum_{i=p}^{n} m_i \boldsymbol{g}^T \frac{\partial T_i}{\partial q_p} {}^i r_i$$

$$= \sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_j} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \sum_{i=p}^{n} m_i \boldsymbol{g}^T \frac{\partial T_i}{\partial q_p} {}^i r_i$$

4.2.3 Forming the Dynamic Equation

- Dynamic Equation of a n-link manipulator:

$$T_i = \sum_{j=i}^{n}\sum_{k=1}^{j} Trace\left(\frac{\partial T_j}{\partial q_k} I_j \frac{\partial T_j^{\ T}}{\partial q_i}\right)\ddot{q}_k + I_{ai}\ddot{q}_i$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} Trace\left(\frac{\partial^2 T_i}{\partial q_k \partial q_m} I_j \frac{\partial T_j^{\ T}}{\partial q_i}\right)\dot{q}_k\dot{q}_m - \sum_{j=1}^{n} m_j \boldsymbol{g}^T \frac{\partial T_i}{\partial q_i}{}^i r_i \qquad (4.23)$$

$$T_i = \sum_{j=1}^{n} D_{ij}\ddot{q}_j + I_{ai}\ddot{q}_i + \sum_{j=1}^{6}\sum_{k=1}^{6} D_{ijk}\dot{q}_j\dot{q}_k + D_i \qquad (4.24)$$

注意：上述惯量项与重力项在机械手的控制中特别重要，它们将直接影响到机械手系统的稳定性和定位精度。只有当机械手高速运动时，向心力和哥氏力才变得重要。

4.3 Summary

- Two methods to form dynamic equation of a rigid body:
  - Lagrangian Equation (Energy-based)
  - Newton-Euler Equation (Force-balance)
- Summarize steps to form Lagrangian Equation of n-link manipulators:
  - Computing the **Velocity** of any given point;
  - Computing total **Kinetic Energy**;
  - Computing total **Potential Energy**;
  - Forming **Lagrangian Function** of the system;
  - Forming Dynamic Equation through **Lagrangian Equation**.

谢 谢！