Stackelberg Game Approach for Robust Optimization With Fuzzy Variables

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***Abstract*—In this article, a new robust optimization method is proposed to simultaneously optimize the expectation and variability of system performance with parametric uncertainties and fuzzy variables. The expectation-entropy model is presented to characterize the fuzzy robust optimization problem as an equivalent biobjective optimization problem. An approximate mapping method is developed to calculate the response of fuzzy variables, which improves the computational efficiency of objective functions. Then, according to the decision makers' preference for objectives, the optimization framework based on Stackelberg game is established. A leader–follower state transition algorithm is designed to search for the equilibrium solutions. Two practical case studies are provided to show the effectiveness of the new optimization approach in both subjective judgment and objective assessment.**

***Index Terms*—Approximate mapping method, fuzzy variable, robust optimization, Stackelberg game, state transition algorithm (STA).**

I. Introduction

In MANY real-world optimization problems, the uncertainties involved are nonprobabilistic in nature. This kind of uncertainty is related to vagueness and imprecision provided by experts in terms of both objective values and subjective judgment [1]–[3]. Handling such uncertainties calls for the use of the fuzzy variable that captures the characteristics of vagueness and imprecision.

The optimization methods with fuzzy variables have been studied in areas such as robust optimization [4]. More specifically, robust optimization theory provides risk-averse methods to find solutions that are insensitive to uncertainties. For unconstrained robust optimization problems, there are two approaches: worst-case robust optimization and realistic robust optimization [5], [6]. The worst-case robust optimization approach guarantees that the value of objective function will never violate the robustness requirements, and the obtained solution can cope with uncertainties for all possible values of parameters. However, the solution is highly conservative to be unrealistic in practical applications. The realistic robust optimization approach establishes a tradeoff model that takes into account both robustness and optimality aiming to improve the average system performance. The solution of the realistic robust optimization approach offers performance that can be close to the optimum for most possible values of uncertain parameters.

The realistic robust optimization approach is effective for many engineering applications [6], and the most common way is to simultaneously optimize the expectation (which can represent optimality) and variability (which can represent robustness) of system performance under uncertain environment [7], [8]. For the fuzzy uncertainty, the expectation-entropy model is proposed to balance the optimality and robustness [9]. The fuzzy expectation is the average value of fuzzy variable in the sense of uncertain measure and represents the size of uncertain variable. The fuzzy entropy is a measure to quantify the uncertainty of fuzzy variable resulting from information deficiency and represents the robustness of an uncertain variable [10]. The calculation of expectation and entropy of the fuzzy system is usually based on the alpha-cut (α-cut) technique, which requires to find the maximum and minimum values of fuzzy response at different cut sets of fuzzy variables. To save the computational cost of finding extreme points, the surrogate model based on Chebyshev polynomials was adopted to predict the fuzzy response [7]. To improve the approximation accuracy, an efficient approximate mapping method based on the polynomial chaos Kriging (PC-Kriging) metamodel is studied and used to calculate the fuzzy expectation and entropy.

When assessing the tradeoff between expectation and entropy in fuzzy robust optimization problems, there exist three highly valued strategies for searching a solution: Pareto-based strategy, Nash-based strategy, and Stackelberg-based strategy [11]. The Pareto-based optimization methods are well known in engineering. For instance, Marano and Quaranta [12] used a multiobjective genetic optimization algorithm to find the Pareto front for the fuzzy-based robust structural design problem. Note that for a Pareto optimal solution, there are three characteristics when searching [13].

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1) The objectives (players' payoff in the game) are equivalent to each other, and there is no objective function dominates others.
2) The objectives (players' payoff in the game) take joint actions to reach a compromise so that the group can achieve its optimal state.
3) The objectives (players' payoff in the game) can communicate with each other before the optimization process.

In practical robust optimization, there exists a strong subjectivity during the decision-making processes. For example, with different market requirement and customer demand, the decision makers may have different preferences for optimality and robustness. In this kind of robust optimization problem, the objectives have a hierarchical relationship, that is, the preferred objective dominates the objective that is without preference. Therefore, the Pareto-based optimization method is not suitable to solve the fuzzy robust optimization problem with preferences.

Nash equilibrium is a noncooperative game, in which each player acts independently, without communication and collaboration. The objectives in Nash-based optimization have equal status [14], and it is difficult to use Nash game strategy to balance objectives with hierarchical relationship.

Stackelberg game is a leader–follower sequential game [15], [16], which is well suited to the hierarchical decision making problem. The leader can anticipate the reactions of follower and optimize his decision accordingly. Thus, the leader stands in a strong position and can achieve more satisfactory results. The Stackelberg game strategy has applied to many areas, such as security domain and manufacturing process [17]. For the optimization problem with multiple goals, the subjective preferences will determine the quality of solutions. In order to better meet the decision maker's preferences, it is reasonable to allocate the leader positions to relatively important goals and the follower positions to other goals. Therefore, to handle the hierarchical relationship between robustness and optimality, the Stackelberg game is investigated in this article to solve the fuzzy robust optimization problem.

The challenge for the fuzzy robust optimization with expectation-entropy model is twofold: 1) how to efficiently calculate the objective functions of fuzzy expectation and entropy, and 2) how to balance the expectation and entropy based on objective assessment and subjective preference. To solve these problems, this article proposes a fuzzy robust optimization method with improvements in objective calculation and optimization framework. The main contribution of this study is as follows.

1) The objective functions of fuzzy expectation and entropy are calculated based on an approximate mapping method, which uses the adaptive PC-Kriging metamodel to efficiently predict the response of fuzzy variables.
2) A Stackelberg-based robust optimization framework is proposed to handle the relationship between robustness and optimality, which can not only objectively address the tradeoff but also subjectively choose priority target, so that it can provide a reasonable optimal robust solution according to the preferences of decision makers.
3) The search of Stackelberg equilibrium solution is solved by a state transition algorithm (STA), which overcomes the nonlinearity and nonconvexity of the objective functions.

The rest of this article is organized as follows. In Section II, the expectation-entropy model for fuzzy robust optimization is established. Section III gives the approximate mapping method for objective function calculation. Section IV provides the framework of the optimization method based on the Stackelberg game. In Section V, two examples are studied to demonstrate the efficiency of the proposed method in Section V. Finally, Section VI concludes this article.

II. Expectation-Entropy Model for Fuzzy Robust Optimization

We consider the optimization problem with fuzzy variable

$$\textbf{Problem P0} \qquad \begin{aligned} &\min_{\boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{p}) \\ &\text{s.t. } \boldsymbol{x}_l \le \boldsymbol{x} \le \boldsymbol{x}_u \end{aligned} \tag{1}$$

where $f(\boldsymbol{x}, \boldsymbol{p})$ is the fuzzy performance function, and the vector $\boldsymbol{x}$ represents decision variable, with $\boldsymbol{x}_l$ and $\boldsymbol{x}_u$ being the lower and upper bounds, respectively. The parameter $\boldsymbol{p}$ is the fuzzy variable, and the membership function is used to represent the degree of uncertainty. The robust optimization method is designed to handle the parameter uncertainties. In this section, Problem **P0** is transformed to a fuzzy robust optimization problem with the expectation-entropy model.

Here, some basic definitions for fuzzy robust optimization based on uncertainty theory are presented [10].

Definition 1: Let β denote a fuzzy variable with assigned membership function μ and r be a real number; then, the concept of possibility $\mathrm{Pos}\{\cdot\}$, necessity $\mathrm{Nec}\{\cdot\}$, and credibility $\mathrm{Cr}\{\cdot\}$ of an event "$\beta \le r$" is defined by

$$\begin{aligned} \mathrm{Pos}\{\beta \le r\} &= \sup_{b \le r} \mu(b) \\ \mathrm{Nec}\{\beta \le r\} &= 1 - \mathrm{Pos}\{\beta > r\} = 1 - \sup_{b > r} \mu(b) \\ \mathrm{Cr}\{\beta \le r\} &= \frac{1}{2}(\mathrm{Pos}\{\beta \le r\} + \mathrm{Nec}\{\beta \le r\}) \\ &= \frac{1}{2}\Big(\sup_{b \le r} \mu(b) + 1 - \sup_{b > r} \mu(b)\Big). \end{aligned}$$

Definition 2: Let β denote a fuzzy variable. The expected value of β is defined by

$$E[\beta] = \int_0^{+\infty} \mathrm{Cr}\{\beta \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\beta \le r\}\mathrm{d}r. \tag{2}$$

Definition 3: Let β denote a fuzzy variable. The entropy of β is defined by

$$\begin{aligned} H[\beta] = &\int_{-\infty}^{+\infty} (-\mathrm{Cr}\{\beta = r\} \ln \mathrm{Cr}\{\beta = r\})\mathrm{d}r \\ &+ \int_{-\infty}^{+\infty} (-(1 - \mathrm{Cr}\{\beta = r\}) \ln(1 - \mathrm{Cr}\{\beta = r\}))\mathrm{d}r. \end{aligned} \tag{3}$$

The fuzzy entropy provides a measure of uncertainty. In general, the entropy of a crisp number is minimum and the entropy of an equipossible fuzzy variable is maximum. Thus, the smaller the fuzzy entropy, the lesser the fuzziness of the variable.

Based on the fuzzy expectation $E[\cdot]$ and fuzzy entropy $H[\cdot]$, respectively, as defined in (2) and (3), the robust model of Problem **P0** with fuzzy variables can be formulated as

$$\textbf{Problem P1} \qquad \begin{aligned} &\min_{\boldsymbol{x}} E[f(\boldsymbol{x},\boldsymbol{p})] \\ &\min_{\boldsymbol{x}} H[f(\boldsymbol{x},\boldsymbol{p})] \\ &\text{s.t.} \boldsymbol{x}_l \leq \boldsymbol{x} \leq \boldsymbol{x}_u. \end{aligned} \tag{4}$$

Remark 1: If the parameter in a function is considered as a fuzzy variable, the response of the function is also a fuzzy variable. Thus, to solve the above problem, the first step is to obtain the membership function of $f(\boldsymbol{x},\boldsymbol{p})$.

Remark 2: Problem **P1** is a biobjective optimization problem. The expected value $E[\cdot]$ can represent the average optimality of the performance function, while the entropy value $H[\cdot]$ can indicate the robustness and variability of the performance function.

III. Approximate Mapping Method for Objective Function Calculation

In this section, we propose an approximate mapping method to compute the objective functions in Problem **P1** based on the adaptive PC-Kriging metamodel. First, the membership function of the performance function value is calculated based on the α-cut technique, which need to obtain the maximum and minimum values of $f(\cdot)$ at different cut sets of the fuzzy variable p. To reduce the computational cost of finding extreme values, an approximate mapping method based on the PC-Kriging metamodel is then used to predict the membership function values. Moreover, based on the analysis of fuzzy set and PC-Kriging model, two learning strategies are investigated to adaptively improve the prediction accuracy.

A. Objective Function Calculation Based on the α-Cut Technique

Based on Zadeh's extension principle [18], [19], the membership function of fuzzy performance $f(\cdot)$ in (4) can be calculated according to the following definition.

Definition 4: Let $\boldsymbol{p} = \{p_1,\ldots,p_n\}$ be n fuzzy variables with membership functions $\mu_{P_1}(p_1),\ldots,\mu_{P_n}(p_n)$; f is the function between $\boldsymbol{p}$ and y such that $y = f(\boldsymbol{p})$. Then, the membership function of y can be defined by

$$\mu_Y(y) = \sup_{(p_1,\ldots,p_n)\in f^{-1}(y)} \min\{\mu_{P_1}(p_1),\ldots,\mu_{P_n}(p_n)\}$$

where f^{-1} is the inverse form of f.

Thus, to find the membership function of y, it is equivalent to find the upper and lower bounds of y at different cut level α ($y^{+\alpha}$ and $y^{-\alpha}$), which can be expressed as

$$\textbf{Problem P2} \qquad \begin{aligned} y^{+\alpha} &= \max_{\boldsymbol{p}} f(\boldsymbol{x},\boldsymbol{p}) \\ y^{-\alpha} &= \min_{\boldsymbol{p}} f(\boldsymbol{x},\boldsymbol{p}) \end{aligned}$$

where $\boldsymbol{p} \in [\boldsymbol{p}^{-\alpha}, \boldsymbol{p}^{+\alpha}]$, and the values of $\boldsymbol{p}^{-\alpha}$ and $\boldsymbol{p}^{+\alpha}$ ($\alpha \in [0,1]$) denote the lower and upper bounds of the fuzzy variable $\boldsymbol{p}$ at the cut level α, respectively.

With limited selected cut level α, the fuzzy response of $y = f(\boldsymbol{x},\boldsymbol{p})$ can be constructed, and the membership function of $f(\boldsymbol{x},\boldsymbol{p})$ can be derived from Problem **P2**. Then, the objective values of expectation and entropy in Problem **P1** can be calculated.

Remark 3: Problem **P2** is nested in Problem **P1**, and it needs to be solved for every candidate solution of $\boldsymbol{x}$.

To decrease the computational cost of the optimization, an approximate mapping method based on the PC-Kriging metamodel is proposed. This method establishes a mapping model from $\boldsymbol{x}$ to $y^{\pm\alpha}$ to estimate the membership function of $f(\boldsymbol{x},\boldsymbol{p})$. A short theoretical background of the PC-Kriging metamodeling approach is presented in the following.

B. Approximate Mapping Method Based on the PC-Kriging Metamodel

The PC-Kriging metamodel [20] is an approximation of the input–output function, which combines the advantages of polynomial chaos expansions and Kriging modeling. The PC-Kriging metamodel uses flexible surface shape to predict complex systems response, and it has been applied in many sensitivity analysis problems and optimization problems [21], [22].

In Problem **P2**, given m samples containing the input ($\boldsymbol{X}$) and output ($\boldsymbol{Y}^\alpha$) as

$$\begin{aligned} \boldsymbol{X} &= [\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m]^T \\ \boldsymbol{Y}^\alpha &= [\boldsymbol{y}^{+\alpha}, \boldsymbol{y}^{-\alpha}]^T \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{y}^{+\alpha} &= [\max f(\boldsymbol{x}_1,\boldsymbol{p}),\ldots,\max f(\boldsymbol{x}_m,\boldsymbol{p})] \\ \boldsymbol{y}^{-\alpha} &= [\min f(\boldsymbol{x}_1,\boldsymbol{p}),\ldots,\min f(\boldsymbol{x}_m,\boldsymbol{p})] \end{aligned}$$

the approximate mapping function between $\boldsymbol{X}$ and $\boldsymbol{Y}^\alpha$ based on the PC-Kriging metamodel can be denoted as

$$\hat{y}^{\pm\alpha} = \sum_{a\in\mathcal{A}} \omega_a \psi_a(\boldsymbol{x}) + Z(\boldsymbol{x}) \tag{5}$$

where $\sum_{a\in\mathcal{A}} \omega_a \psi_a(\boldsymbol{x})$ corresponds to a polynomial chaos expansion with orthonormal polynomials $\psi_a(\boldsymbol{x})$ and coefficients ω_a, which describes the trend of the PC-Kriging model; the function $Z(\boldsymbol{x})$ is a realization of Gaussian process, which represents local deviations. The PC-Kriging model can be interpreted as a universal Kriging model with a specific trend.

Based on the PC-Kriging metamodel in (5), the nested double-loop optimization structure is simplified to a single-loop optimization problem. As the PC-Kriging metamodel contains stochastic process, the output results include the predicted value $\hat{y}^{\pm\alpha}$ and the prediction variance $\sigma^2_{\hat{y}^{\pm\alpha}}$. This variance can quantify the uncertainty of the prediction, which can be analyzed for improving the model design.

C. Adaptive Learning Strategy

To improve the approximation accuracy of the mapping method, two learning strategies are proposed based on the property of fuzzy set and PC-Kriging model.

1) Learning Strategy Based on Interval Limitation: According to the theory of fuzzy set, for fuzzy variable p with any two cut levels $\alpha_1, \alpha_2 \in [0,1]$, if $\alpha_1 \leq \alpha_2$, then the cut sets $[p^{-\alpha_2}, p^{+\alpha_2}] \subseteq [p^{-\alpha_1}, p^{+\alpha_1}]$. Thus, in Problem **P2**, the following is true:

$$\alpha_1 \leq \alpha_2 \Rightarrow y^{+\alpha_1} \geq y^{+\alpha_2} \text{ and } y^{-\alpha_1} \leq y^{-\alpha_2}. \tag{6}$$

Therefore, to evaluate the approximate value, the steps of the learning strategy based on interval limitation are as follows. First, we calculate the value of y at the cut level of $\alpha = 1$ because it is a fixed number. Then, by gradually decreasing the value of α, the predicted value $\hat{y}^{\pm\alpha}$ is checked according to the limitation in (6). If the predicted value does not satisfy the limitation, it means that the PC-Kriging metamodel cannot provide a convincing membership function value, and it is necessary to perform calculations using an accurate method (such as an optimization method). After obtaining the accurate values, the pair of data ($\boldsymbol{x}$ and $y^{\pm\alpha}$) is considered as a new sample that can be used to update the PC-Kriging metamodel. If the predicted value satisfies the limitation, the following strategy is used for further evaluation.

2) Learning Strategy Based on Variation Limitation: According to the stochastic property of PC-Kriging, the second strategy aims to identify the potentially "dangerous" point, whose PC-Kriging variance is high. Therefore, a learning function U is studied, which is defined as the ratio between the predicted value and the prediction variance of Kriging [23], [24]:

$$U(\boldsymbol{x}) = \frac{|\hat{y}^{\pm\alpha}(\boldsymbol{x})|}{\sigma_{\hat{y}^{\pm\alpha}}(\boldsymbol{x})}. \tag{7}$$

Based on the analysis and validation in [23], we consider that the condition for $U(\boldsymbol{x}) \geq 2$ can guarantee great accuracy of the prediction model. Once a point with $U(\boldsymbol{x}) < 2$, an accurate method will be performed to calculate the value of $y^{\pm\alpha}$, and the pair of data ($\boldsymbol{x}$ and $y^{\pm\alpha}$) will be used to update the PC-Kriging metamodel.

With the above two learning strategies, the sample points of the PC-Kriging metamodel will be close to the failure region, and the approximation accuracy can be adaptively improved. The flowchart of the approximate mapping method based on the adaptive PC-Kriging metamodel is shown in Fig. 1.

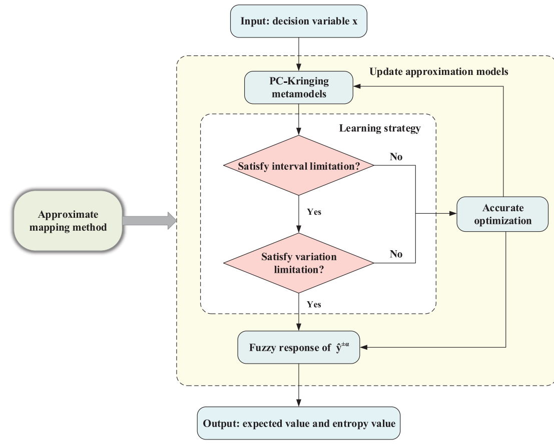

Fig. 1. Flowchart of the approximate mapping method based on the adaptive PC-Kriging metamodel.

IV. Optimization Method Based on Stackelberg Game

Considering the hierarchy between optimality ($E[\cdot]$) and robustness ($H[\cdot]$) caused by the preferences of decision makers, the robust optimization of Problem **P1** first needs to satisfy the higher level objective and then optimize the lower level objective. Stackelberg game [25], as a hierarchical game, consists of two groups of players: leaders and followers. Since it is hard to balance the optimality and robustness in robust optimization problems, the Stackelberg strategy is adopted to provide higher priority to the leader players based on the decision makers' preference. Therefore, the robust optimization method based on the Stackelberg game is introduced to solve Problem **P1** in (4).

A. Stackelberg Game

The mathematical description for the Stackelberg game is as follows. Let J_u and J_v be the payoff function of the leader and the follower, respectively. The search space of decision variable ($\boldsymbol{x} \in S$) consists of the leader's search space ($\boldsymbol{x}_u \in S_u$) and the follower's search space ($\boldsymbol{x}_v \in S_v$). We assume that each player (J_u and J_v) controls its corresponding variable ($\boldsymbol{x}_u$ and $\boldsymbol{x}_v$). The multiobjective optimization problem based on the Stackelberg game [26] is defined by

Problem P3

$$\begin{aligned} &\text{Leader: } \min_{\boldsymbol{x}_u \in S_u} J_u(\boldsymbol{x}_u, \boldsymbol{x}_v) \\ &\text{Follower: } \min_{\boldsymbol{x}_v \in S_v} J_v(\boldsymbol{x}_u, \boldsymbol{x}_v). \end{aligned} \tag{8}$$

Let vector $\boldsymbol{x}^* = (\boldsymbol{x}_u^*, \boldsymbol{x}_v^*)$ be a Stackelberg equilibrium; then

$$J_u(x_u^*, x_v^*) = \inf_{x_u} J_u(x_u, x_v^*(x_u))$$

where $x_v(x_u)$ is the reaction function coming from the solution of $\min_{\boldsymbol{x}_v} J_v(\boldsymbol{x}_u, \boldsymbol{x}_v)$:

$$J_v(x_u^*, x_v^*) = \inf_{x_v} J_v(x_u^*, x_v).$$

Both the leader and the follower can find their own optimal solutions via solving

$$\left.\frac{DJ_u}{D\boldsymbol{x}_u}\right|_{(x_u^*, x_v^*)} = \left.\frac{\partial J_u}{\partial \boldsymbol{x}_u}\right|_{(x_u^*, x_v^*)} + \left.\frac{\partial J_u}{\partial \boldsymbol{x}_v}\frac{\partial \boldsymbol{x}_v}{\partial \boldsymbol{x}_u}\right|_{(x_u^*, x_v^*)} = 0 \tag{9}$$

and

$$\left.\frac{\partial J_v}{\partial \boldsymbol{x}_v}\right|_{(x_u^*, x_v^*)} = 0. \tag{10}$$

The information flow diagram of the Stackelberg game is shown in Fig. 2. The leader provides the information of $\boldsymbol{x}_u$ to the follower and receives the reaction function $x_v(x_u)$ of the follower with respect to the leader's decision [17]. Thus, the

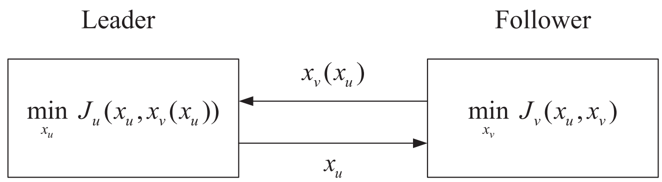

Fig. 2. Information flow diagram of Stackelberg game.

leader can take advantage of its leadership position to achieve better performance.

Remark 4: Based on the Stackelberg game strategy, Problem **P1** in (4) can be transformed to Problem **P3** in (8) with different preferences for robustness and optimality.

B. State Transition Algorithm

To further solve the optimization problem for the leader and the follower, a global evolutionary algorithm named STA [27], [28] is investigated to generate optimal solutions. The STA has strong search ability in both global and local spaces, and it has been applied to many engineering design applications [29]–[32].

Based on the framework of STA, there are four transformation operators to generate candidate solutions.

Rotation transformation:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \eta_1 \frac{1}{\|\boldsymbol{x}_k\|_2} R_1 \boldsymbol{x}_k. \tag{11}$$

Translation transformation:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \eta_2 R_2 \frac{\boldsymbol{x}_k - \boldsymbol{x}_{k-1}}{\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\|_2}. \tag{12}$$

Expansion transformation:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \eta_3 R_3 \boldsymbol{x}_k. \tag{13}$$

Axesion transformation:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \eta_4 R_4 \boldsymbol{x}_k. \tag{14}$$

Here, $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$ represent the solution at the kth and (k+1)th generation, respectively; parameters η_1, η_2, η_3, and η_4 are positive constants representing rotation factor, translation factor, expansion factor, and axesion factor, respectively; elements in matrices R_1, R_2, R_3, and R_4 are designed based on the function of operators. In order to find the global optima, the transformation operators in (11)–(14) have the functions of local search, line search, global search, and single dimensional search, respectively.

The structure of STA is shown in Algorithm 1.

As for further detailed explanation, Algorithm 2 illustrates the process of expansion($\cdot$).

Algorithm 1: Pseudocode of STA.

Input:
iter$_{\max}$: maximum number of iterations
SE: search enforcement
S: search space of decision variable
Output:
Best$*$: optimal solution

```
State ← initialization(SE,S)
Best ← fitness(funfcn,State)
For k = 1 to iter_max do
   if η1 < η1,min then
      η1 ← η1,max
   end if
   Best ← expansion(funfcn,Best,SE,η2, η3)
   Best ← rotation(funfcn,Best,SE,η2, η1)
   Best ← axesion(funfcn,Best,SE,η2, η4)
   η1 ← η1 · fc^(−1)
end for
Best* ← Best
```

Algorithm 2: Pseudocode of Expansion($\cdot$).

Input:
oldBest: the best solution obtained by the last transformation
Output:
Best: the best solution

```
oldfBest ← fevel(funfcn,oldBest)
State ← op_expand(oldBest,SE,η3)
Selecting the best solution based on the "greedy
  criterion".
[newBest,newfBest] ← fitness(funfcn,State)
if newfBest < fBestthen
   fBest ← newfBest
   Best ← newBest
   State ← op_translate(oldBest,newBest,SE,η2)
   Selecting the best solution based on the "greedy
     criterion".
   [newBest,newfBest] ← fitness(funfcn,State)
   if newfBest < fBestthen
      fBest ← newfBest
      Best ← newBest
   end if
else
   Best ← oldBest
end if
```

The process of for rotation($\cdot$) and axesion($\cdot$) is similar to the procedure in Algorithm 2. As shown in the above algorithms, procedure initialization($\cdot$) randomly generates *SE* samples in the search space and selects the best one as the initial solution. Rotation factor η_1 is decreasing periodically from $\eta_{1,\max}$ to $\eta_{1,\min}$ to change the local search range. *SE* is the search enforcement index to indicate the number of candidate solutions generated by each operator. Procedures op_expand($\cdot$) and op_translate($\cdot$) represent the implementations of searching candidates using the rotation and the translation operators. Procedure fitness($\cdot$) represents the implementation of selecting the best solution from *SE* candidates based on the "greedy criterion," which means that the optimal solution will always consider a solution with a better objective function value [28]. In addition, the translation transformation is only performed when a solution that is better than the incumbent

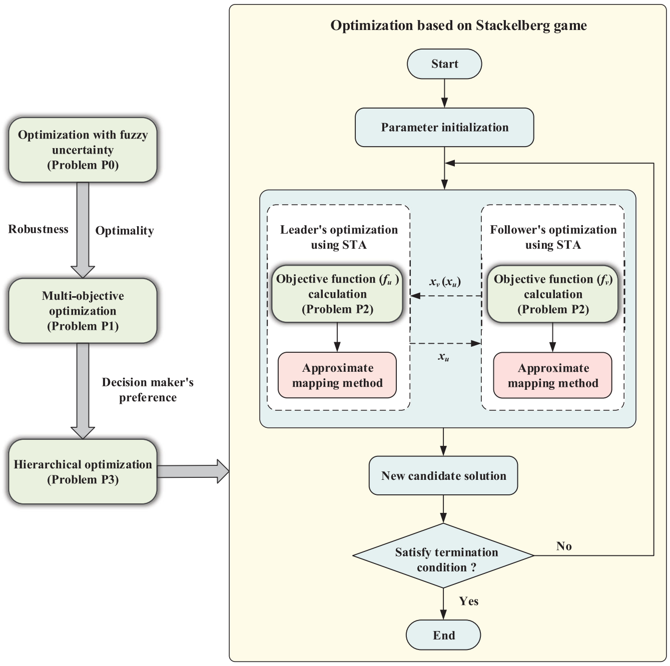

Fig. 3. Optimization method based on Stackelberg game.

best solution can be found by rotation, expansion, or axesion transformation.

C. Optimization Framework

The complete framework of the optimization method based on the Stackelberg game is shown in Fig. 3. The follower's optimization problem is first solved according to the STA. Then, the leader's optimization process is conducted, which takes into account the reaction function of $\boldsymbol{x}_v(\boldsymbol{x}_u)$. The optimization process is alternatively performed until the termination condition is met, i.e., the maximum number of iterations is reached.

It is appeared in many studies [17], [33] that the reaction function $\boldsymbol{x}_v(\boldsymbol{x}_u)$ can be approximated as

$$\boldsymbol{x}_v^{k+1} = \boldsymbol{x}_v^k + \frac{\partial \boldsymbol{x}_v}{\partial \boldsymbol{x}_u}(\boldsymbol{x}_u^{k+1} - \boldsymbol{x}_u^k)$$

where

$$\frac{\partial \boldsymbol{x}_v}{\partial \boldsymbol{x}_u} = \lim_{\triangle \boldsymbol{x}_u \to 0} \frac{\triangle \boldsymbol{x}_v}{\triangle \boldsymbol{x}_u} = \lim_{\triangle \boldsymbol{x}_u \to 0} \frac{\boldsymbol{x}_v(\boldsymbol{x}_u + \triangle \boldsymbol{x}_u) - \boldsymbol{x}_v(\boldsymbol{x}_u)}{\triangle \boldsymbol{x}_u}.$$

Note that if the relationship between $\boldsymbol{x}_v$ and $\boldsymbol{x}_u$ is complex with nonlinear and nonconvex properties, the large variation of the leader's decision variable may cause large errors in the anticipation of the follower's reaction information. In order to better capture the reaction information and then improve the convergence performance in the leader's optimization and follower's optimization, the PC-Kriging metamodel with the learning function U in (7) introduced in Section III is adopted to approximate the reaction function.

Obviously, the solution obtained by the method in Fig. 3 satisfies the property of Stackelberg equilibrium in (9) and (10). At a Stackelberg equilibrium, the leader's objective reaches optimal point in consideration of the follower's reaction, and the follower's objective reaches the optimal point according to the leader's decision.

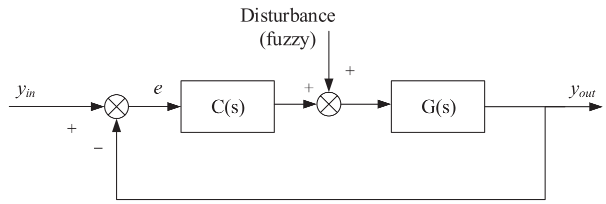

Fig. 4. Closed-loop control system.

With the above optimization method shown in Fig. 3, if we consider the expected value as a leader and the entropy value as a follower, Problem **P1** in (4) can be solved by prioritizing to optimality, and vice versa.

V. Examples and Results

In this section, the feasibility and effectiveness of the robust-fuzzy-optimization-method-based Stackelberg game are verified via two applications: 1) proportional-integral-derivative (PID) controller and 2) blending process.

We use the optimization framework shown in Fig. 3. For the optimization of each player using the STA method, its parameter settings are the same as the previous paper [27]–[29], which all successfully solve the optimization problems such as benchmarks, image segmentation, and clustering problem: $\eta_{1,\max} = 1, \eta_{1,\min} = 10^{-4}, \eta_2 = 1, \eta_3 = 1, \eta_4 = 1, fc = 2, SE = 30$. The maximum number of iterations $\text{iter}_{\max}$ depends on the complexity of the problem. In this article, ten generations may be sufficient for each player to ensure the convergence performance of the STA method. For the Stackelberg game strategy, the maximum number of iterations is set to 10, and the assignment of the decision variables for the leader objective and the follower objective is designed based on their space distance [34]. For the approximate mapping method used for computing the objective function values, the parameter settings of the adaptive PC-Kriging metamodel are the same as those in UQLab [35]. The initial number of sample points to construct the PC-Kriging metamodels is set to $50q$, where q is the total number of decision variables. All results are obtained by MATLAB R2017b software.

A. Case 1: Optimization for PID Controller

PID control is the most common control technique used in practical applications. To deal with the parametric uncertainties in the PID control process, the controller should make a tradeoff between the robustness of the response to uncertainty and the response speed. Therefore, we study the optimization problem for the PID controller with parametric uncertainties to illustrate the effectiveness of the proposed Stackelberg-based robust optimization framework.

Consider the closed-loop control system in Fig. 4, where module C represents the controller and module G is the process subject to parameter uncertainty caused by fuzzy disturbance.

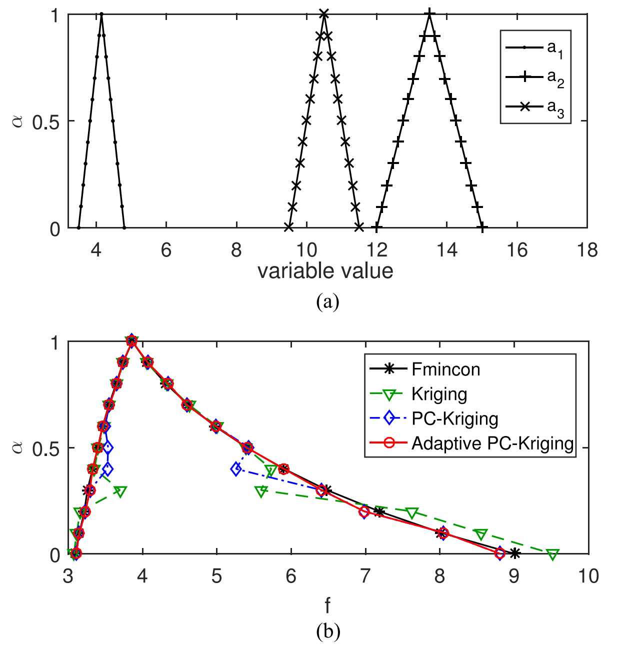

Fig. 5. Membership function of fuzzy variables. (a) Parameter variables a_1, a_2, and a_3. (b) Function $f(a_1, a_2, a_3)$.

TABLE I
COMPARISON OF APPROXIMATION ACCURACY

Method	Mean Squared Error	R-squared Score
Kriging Metamodel	0.08003	0.9707
PC-Kriging Metamodel	0.0256	0.9907
Adaptive PC-Kriging Metamodel	0.0040	0.9986

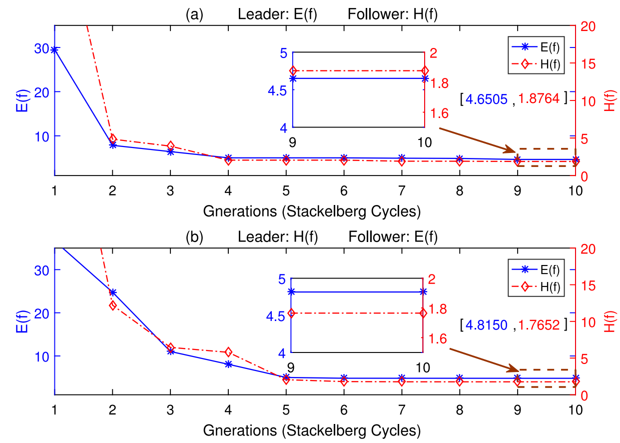

Fig. 6. (a) and (b) Convergent trajectories of the robust optimization procedure based on the Stackelberg game.

Signals y_{in}, y_{out}, and e are the input, output, and tracking error of the system, respectively. The controller C and the process G take the forms [36]

$$C(s) = K_p + \frac{K_i}{s} + K_d s,\ \ G(s) = \frac{5.2(s+2)}{s(s^3+a_2 s^2+a_1 s+a_3)}$$

where K_p, K_i, and K_d represent the proportional, integral, and differential gains, respectively; parameters a_1, a_2, and a_3 are triangular fuzzy variables

$$a_1 = (3.5, 4.15, 4.8), a_2 = (12, 13.5, 15), a_3 = (9.5, 10.5, 11.5).$$

Based on the integral time absolute error criteria, the biobjective optimization problem for the PID controller is represented as

$$\min_{K_p, K_i, K_d} \ E[f(K_p, K_i, K_d, a_1, a_2, a_3)]$$
$$\min_{K_p, K_i, K_d} \ H[f(K_p, K_i, K_d, a_1, a_2, a_3)]$$

where

$$f(K_p, K_i, K_d, a_1, a_2, a_3) = \int_0^\infty t|e(t)|dt.$$

To demonstrate the effectiveness of the approximate mapping method, we compare the approximation results of the adaptive PC-Kriging metamodel with that of other three methods (including MATLAB optimization tool `fmincon`, the surrogate model based on the Kriging metamodel, and the PC-Kriging metamodel). The initial sample numbers for establishing metamodels are all set to 150.

Fig. 5 shows the membership functions of a_1, a_2, a_3, and f (with $K_p = 1.9011$, $K_i = 0$, and $K_d = 1.0938$). To capture the character of the membership function of f, we select 11 cut levels ($\alpha = 0, 0.1, 0.2, \ldots, 1$). Since the optimization tool `fmincon` can find the minimum of nonlinear multivariable function, the results obtained by `fmincon` are as the real membership function value, and the Kriging metamodel, the PC-Kriging metamodel, and the adaptive Kriging metamodel are used to approximate the real value.

It can be seen that the results of the adaptive PC-Kriging metamodel are closer to the real value obtained by `fmincon`. The detailed approximation accuracies of the 21 points based on the adaptive PC-Kriging metamodel, the PC-Kriging metamodel, and the Kriging metamodel are listed in Table I. In Table I, mean squared error and R-squared score are two typical evaluation indices [37]. The smaller the mean square error and the larger the R-square score, the better the fitting performance. Thus, the approximate mapping method based on the adaptive PC-Kriging metamodel is more effective to compute the membership function of f.

The convergent trajectories of the robust optimization procedure based on the Stackelberg game are shown in Fig. 6. After ten generations of the Stackelberg-based optimization, the expected value (represented the average performance) and the entropy value (represented the stability performance) can converge to an equilibrium point. The final optimal solution shown in Fig. 6(b) has better entropy value than that in Fig. 6(a), because the leader objective in Fig. 6(b) is the entropy value. Based on the Stackelberg game strategy, the leader can achieve higher satisfaction than the follower. Thus, the results shown in Fig. 6 can provide alternatives to decision makers according to their preference for optimality or robustness.

To verify the effectiveness of the proposed robust optimization method, we compare the performance of the PID controller

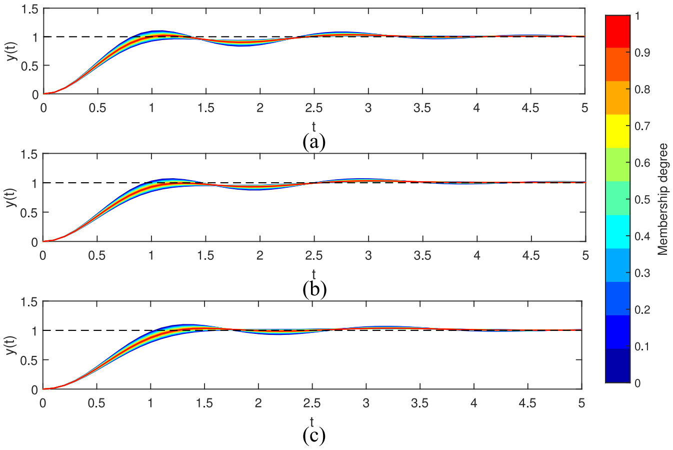

Fig. 7. (a)–(c) Twenty-one possible responses of the uncertain system.

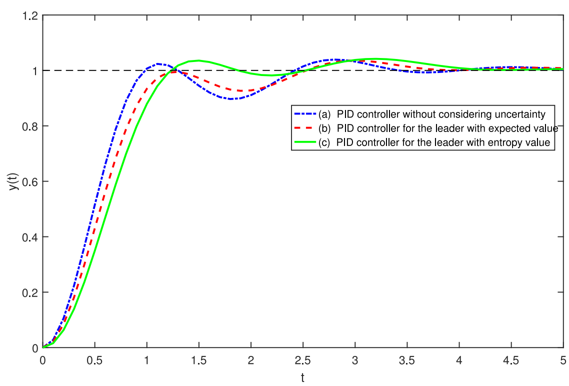

Fig. 8. Mean performance of three PID controllers.

optimized by the Stackelberg-based robust method and the deterministic method. Fig. 7 shows the 21 possible responses by three different PID controllers, which include the PID controller with a preference for expected value (a), the PID controller with a preference for entropy value (b), and the PID controller without consideration of uncertainty (c). The 21 possible responses correspond to the upper and lower bounds of 11 membership degrees ($\alpha = 0, 0.1, 0.2, \ldots, 1$). It is worth noting that the upper and lower bounds of the response with a membership of 1 are the same. The colors in Fig. 7(a)– (c) represent the membership value. It can be observed that the outputs with high membership values have better performance than that with low membership values, which indicates that the system without uncertainty can achieve the optimal performance. With the fuzzy possibility increases, the output performance will deteriorate.

The mean performance of these three controllers (shown in Fig. 7) is displayed in Fig. 8. For the leader objective with fuzzy expectation, the closed-loop response is faster; whereas for the leader objective with fuzzy entropy, the closed-loop response is more robust.

The detailed analysis of the PID response performance is given in Table II. Compared with the controller without considering the uncertainty, the PID controllers based on robust optimization have more stable performance, whose maximum settling time are shorter, maximum overshoots are smaller, and

TABLE II
COMPARISON BETWEEN THE PROPOSED ROBUST OPTIMIZATION METHOD AND THE DETERMINISTIC OPTIMIZATION METHOD

Methods		Optimization without Considering Uncertainty	Proposed Robust Optimization	
			Leader: $E(f)$	Leader: $H(f)$
Objectives	E(f)	4.5651	4.6505	4.8150
	H(f)	2.4905	1.8764	1.7652
Control Performance	Rise Time	[0.56,0.78]	[0.64,0.90]	[0.68,0.94]
	Settling Time	[3.32,5.52]	[3.84,5.42]	[3.58,5.14]
	Overshoot	[1.66%,10.57%]	[1.99%,6.85%]	[3.15%,10.01%]
	Steady-state error	[2.2866E-6,4.4963E-4]	[2.6410E-4,3.6676E-4]	[6.7863E-7,3.5795E-4]

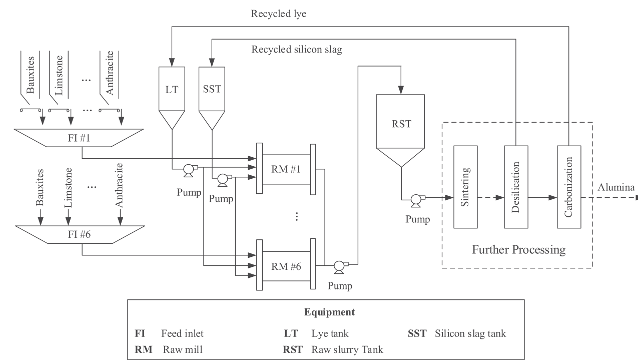

Fig. 9. Blending process of alumina production.

maximum steady-state errors are smaller. The PID performance of the controller without considering the uncertainty has a shorter rise time, because it only focuses on the performance of the deterministic system. When uncertainty occurs, the stability of the system cannot be guaranteed. As for the PID controllers based on robust optimization, if the leader objective is fuzzy expectation, the PID outputs have shorter rise time and smaller overshoots; if the leader objective is the fuzzy entropy, the PID outputs have shorter settling time and smaller steady-state error. Therefore, based on the proposed optimization method, the system can achieve robust performance under different preferences.

B. Case 2: Optimization for Blending Process

The blending process is the first step of nonferrous metallurgical process, which aims to blend and grind different raw materials and meet the production requirement [38], [39]. The ingredients of the raw materials usually change as the environment or the operations change. We investigate the optimization of the blending process to verify the feasibility of the proposed Stackelberg-based robust optimization method in complex industrial processes.

Take the alumina production (shown in Fig. 9) as an example. The raw materials (including bauxite with different grades, limestone, alkali powder, etc.) are fed into raw mills and ground wetly to form the raw slurry. After the blending process, the raw slurry is pumped into tanks for further processing. The silicon slag and the lye produced by desilication and carbonization are recycled as raw materials, respectively.

The quality of raw slurry depends on the composition of CaO, Na_2O, SiO_2, Fe_2O_3, and Al_2O_3. There are three indices related to the proportions of these oxides, which can be expressed as follows.

Alkali standard:

$$I_1 = \frac{1.645 \times w(\mathrm{Na_2O})}{w(\mathrm{Al_2O_3}) + 0.6375 \times w(\mathrm{Fe_2O_3})}. \tag{15}$$

Calcium standard:

$$I_2 = \frac{1.071 \times w(\mathrm{CaO})}{w(\mathrm{SiO_2})}. \tag{16}$$

Aluminum standard:

$$I_3 = \frac{w(\mathrm{Al_2O_3})}{w(\mathrm{SiO_2})}. \tag{17}$$

Here, $w(\cdot)$ represents the mass of oxides, I_1 is the index for the mole ratio of Na_2O to Al_2O_3 and Fe_2O_3, I_2 is for the mole ratio of CaO to SiO_2, and I_3 denotes the mass ratio of Al_2O_3 to SiO_2.

Supposing the number of the raw materials is M, the mass equations of CaO, Na_2O, SiO_2, Fe_2O_3, and Al_2O_3 can be described based on the theory of mass balance

$$w(\mathrm{CaO}) = \rho_1 p_{C,1} x_1 + \sum_{i=2}^{M} p_{C,i} x_i$$

$$w(\mathrm{Na_2O}) = \rho_1 p_{N,1} x_1 + \sum_{i=2}^{M} p_{N,i} x_i$$

$$w(\mathrm{SiO_2}) = \rho_1 p_{S,1} x_1 + \sum_{i=2}^{M} p_{S,i} x_i$$

$$w(\mathrm{Fe_2O_3}) = \rho_1 p_{F,1} x_1 + \sum_{i=2}^{M} p_{F,i} x_i$$

$$w(\mathrm{Al_2O_3}) = \rho_1 p_{A,1} x_1 + \sum_{i=2}^{M} p_{A,i} x_i$$

where $p_{C,i}, p_{N,i}, p_{S,i}, p_{F,i}$, and $p_{A,i}$ represent the proportions of CaO, Na_2O, SiO_2, Fe_2O_3, and Al_2O_3 in the ith raw material, respectively; the variable x_i denotes the feed rate of the ith raw material, and x_1 and x_2 are the feed rates of silicon slag and lye, respectively; and the parameter ρ_1 means the density of silicon slag.

To satisfy the production requirements of further processing, the quality indices of raw slurry need to within certain ranges. In fact, the targets of quality indices are more suitable to be described as fuzzy variables rather than bounded variables. According to the operators' experience, the indices should preferably reach the optimal value, while it is also acceptable if the indices are within certain ranges. Moreover, the proportions of the oxides in raw materials ($\boldsymbol{p} = [p_C, p_N, p_S, p_F, p_A]$) are also uncertain variables due to the diversification of mine sources. Since it is difficult to complete the real-time measurement of the information of $\boldsymbol{p}$, the predictive model is used to estimate the ranges of the proportions. Thus, the parameter $\boldsymbol{p}$ is considered as a fuzzy variable based on expert's prediction. Therefore, the robust optimization model is formulated as

$$\min_{\boldsymbol{x}} \; E[f(\boldsymbol{x}, \boldsymbol{p})]$$
$$\min_{\boldsymbol{x}} \; H[f(\boldsymbol{x}, \boldsymbol{p})]$$

TABLE III
PREDICTED NOMINAL PROPORTIONS OF THE OXIDES IN RAW MATERIALS

Raw Materials	CaO	Na_2O	SiO_2	Fe_2O_3	Al_2O_3
Bauxite (high)	2.24%	0.50%	8.83%	7.01%	64.40%
Bauxite (normal)	3.20%	0.42%	9.27%	8.11%	64.20%
Bauxite (low)	2.80%	0.40%	10.90%	17.08%	53.60%
Bauxite (iron)	3.00%	0.46%	10.60%	5.91%	64.10%
Limestone	95.30%	0.10%	4.55%	0.44%	1.50%
Alkali	–	98.70%	–	–	–
Anthracite	–	–	7.14%	0.89%	4.93%
Silicon slag	14.45%	17.12%	14.00%	1.76%	26.86%
Lye	–	287.60 g/L	–	–	70.27 g/L

1) "–" means that the content of the oxide is relatively low without the need for testing and analysis. 2) The composition analysis results of lye are the concentrations of Na^{2+} and Al^{3+}, so the detection results are converted into the concentration of corresponding oxides (Na_2O and Al_2O_3) by chemical analysis.

where the performance function is the sum of squared errors of the three indices and their target values. Considering that these three indices are equally important for the quality of the raw slurry in the practical alumina production process, the function $f(\cdot)$ is defined as

$$f(\boldsymbol{x}, \boldsymbol{p}) = (I_1 - \hat{I}_1)^2 + (I_2 - \hat{I}_2)^2 + (I_3 - \hat{I}_3)^2$$

the target values of quality indices $\hat{I}_1, \hat{I}_2, \hat{I}_3$ and the proportion of oxides in raw materials $\boldsymbol{p}$ are fuzzy variables.

In this article, an alumina smeltery in China is investigated as a case study. There are nine kinds of raw materials used to produce the raw slurry, which include four kinds of bauxite (high grade, normal grade, low grade, and iron bauxite), two kinds of returned raw materials (silicon slag and lye), and other three auxiliary raw materials (limestone, alkali, and anthracite). Thus, the robust optimization problem for blending process has nine decision variables and 48 fuzzy uncertain parameters (45 component parameters and three indicator parameters).

An example of the predicted nominal proportions of the oxides in several raw materials is listed in Table III. The ideal quality indices are $[\hat{I}_1, \hat{I}_2, \hat{I}_3] = [1.05, 1.95, 4.75]$. All these uncertain parameters are considered as triangular fuzzy variables.

Based on the proposed robust optimization method, the quality indices of 41 samples are shown in Figs. 11 and 12. The 41 possible responses correspond to the upper and lower bounds of 21 membership degrees ($\alpha = 0, 0.05, 0.1, \ldots, 1$). In addition, Fig. 10 shows the quality indices under the optimal operation without considering uncertainty. The colors in Figs. 10–12 represent the membership value of fuzzy parameters. The acceptable ranges of quality indices are $I_1 = [1.03, 1.07], I_2 = [1.88, 2.02]$, and $I_3 = [4.65, 4.85]$.

From Figs. 10–12, we can observe that the red points are around the ideal values, while the blue points are near the boundary of the acceptable ranges, or even beyond the acceptable range. This is because with the increase of parameter fuzziness, it is more and more difficult to guarantee the quality index. Moreover, the possibility of violating the requirement for index I_1 is less than that for indices I_2 and I_3, and this result is related to the mechanism model of the quality indices. From (15), I_1 represents the alkali standard, and its value relates to the

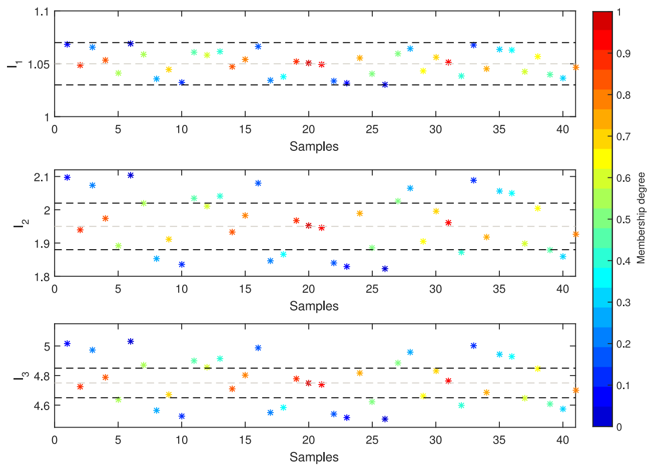

Fig. 10. Quality indices without considering uncertainty.

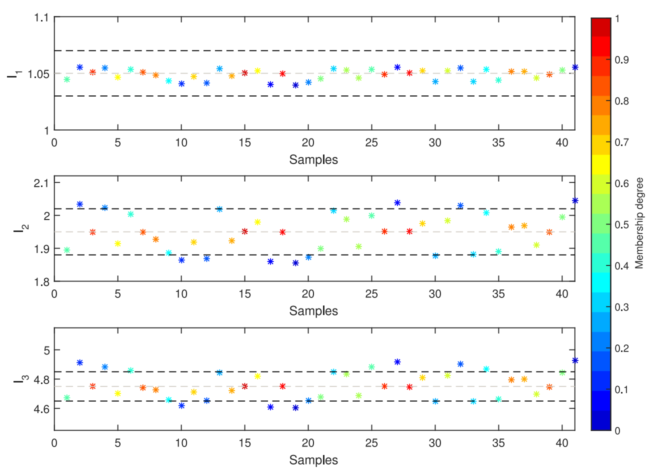

Fig. 11. Quality indices for the leader with the expected value.

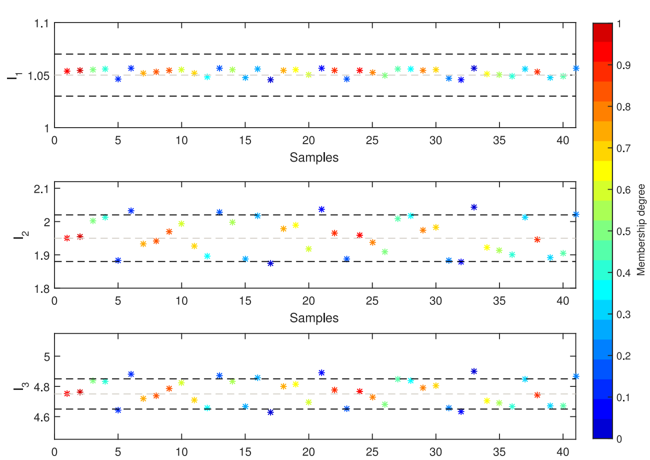

Fig. 12. Quality indices for the leader with the entropy value.

TABLE IV
COMPARISON BETWEEN THE PROPOSED METHOD AND THE OPTIMAL OPERATION

Methods		Optimal Operation without considering uncertainty	Proposed Method Leader: $E(f)$	Proposed Method Leader: $H(f)$
Objectives	E(f)	0.0296	**0.0144**	0.0148
	H(f)	0.0697	0.0343	**0.0339**
Quality Indices	I_1	[1.0306,1.0693]	[1.0399,1.0557]	[1.0452,1.0557]
	I_2	[1.8223,2.1047]	[1.8560 2.0449]	[1.8756,2.0424]
	I_3	[4.5035,5.0324]	[4.6024 4.9267]	[4.6275,4.8980]
Qualified Rates	I_1	100%	100%	100%
	I_2	53.66%	73.17%	82.93%
	I_3	41.46%	68.29%	78.05%

The qualified rate is the percentage of quality indices within the acceptable ranges.

mass of Na_2O, Al_2O_3, and Fe_2O_3. According to Table III, the main components in the lye are Na_2O and Al_2O_3, and the main component in the alkali is Na_2O. Thus, the fluctuation of index I_1 is mainly affected by the uncertainties in the lye and alkali. From (16) and (17), indices I_2 and I_3 represent calcium standard and aluminum standard, respectively. Their values relate to the mass of CaO and SiO_2. According to Table III, most of the raw materials contain these two oxides. Thus, when the proportions of the oxides in raw materials become uncertain, the value of indices I_2 and I_3 will easily be affected.

Compared with the optimal operation without considering uncertainty, the proposed robust optimization method has better optimization results in terms of average performance and variability performance. For the leader with expected value, the red points (with high possibility of occurrence) are closer to the ideal value, while for the leader with entropy value, the blue points (with low possibility of occurrence) slightly violate the acceptable range.

In order to further analyze the results in the above three figures, Table IV provides the detail comparison results between the proposed method and the deterministic method. It can be observed that the Stackelberg game strategy can provide results with the preference for optimality or robustness. Compared with the optimal operation without considering uncertainty, the results under the proposed method have smaller variations and higher qualified rates (more than 65%).

VI. CONCLUSION

This article presented a new framework to solve the fuzzy robust optimization problems. The framework novelty was as follows. First, the approximate mapping method (based on the adaptive PC-Kriging metamodel) could effectively approximate the performance function of fuzzy variables. Second, the Stackelberg game strategy could successfully balance the robustness and optimality based on the decision makers' preferences. The STA searched promising candidates and selected the global optimal solution for Stackelberg equilibrium. The Stackelberg-based robust optimization framework was used to study the optimization of the PID controller and the optimization of the blending process. Simulation results show that the proposed robust optimization method could provide solutions with different objective preferences.

In the future, we plan to study the robust optimization methods for constrained fuzzy design problems. The selection strategy will be developed to balance more attributes of optimization performance, and the optimization algorithm will be further analyzed to improve the search ability of the STA method.

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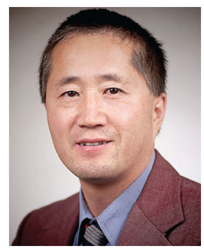

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