Dynamic stiffness method and CUF-based component-wise theories applied to free vibration analysis of solid beams, thin-walled structures and reinforced panels

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1. Introduction

Beam models are widely used to analyse the mechanical behaviour of slender bodies, such as columns, rotor-blades, aircraft wings, towers, antennae and bridges amongst others. Interest in beam models is mainly due to their simplicity and low computational costs when compared to 2D (plate/shell) or 3D (solid) models. The free vibration analysis of beam structures has always been a major area of activity in structural design. The results of modal analyses are, in fact, of great interest in dynamic response analyses, acoustics, aeroelasticity and also to avoid resonance. The classical and well-known beam theories in the free vibration analysis of beam structures are those by Euler [1] and hereinafter referred to as EBBM, and Timoshenko [2,3] and hereinafter referred to as TBM. The former does not account for transverse shear distribution along the cross-section of the beam together with the effects of rotatory inertia. These models yield reasonably good results when slender, solid section, homogeneous structures are subjected to flexure. Conversely, the analysis of deep, thin-walled, open section beams may require more sophisticated theories to achieve sufficiently accurate results [4,5]. Therefore, it is necessary to develop a higher-order beam model and an accurate and efficient analysis method for free vibration of beams in engineering applications.

Over the last century, a number of refined beam theories have been developed to overcome the limitations of classical beam modelling. Different approaches have been used to improve the beam models [6–13], which include the introduction of shear correction factors, the use of warping functions based on de Saint-Venant’s solution, the variational asymptotic solution (VABS), the generalized beam theory...
A comprehensive review of existing beam theories was published by Kapania and Raciti [6,7]. Another review of modern theories for beam structures was published by Carrera et al. [8]. As far as the free vibration analysis is concerned, a brief overview about refined 1D models is given here for the sake of completeness. Early researchers have focussed on the use of appropriate shear correction factors to increase the accuracy of classical 1D formulations, notable contributions include the works of Timoshenko and Goodier [9], Sokolniko [14], Stephen [15], and Hutchinson [16]. The shear correction factor has generally been used as a static concept which is restrictive. In this regard, Jensen [17] demonstrated that the shear correction factor can exhibit variations with respect to the natural frequencies. Additionally, a review article by Kaneko [18] and a publication by Dong et al. [19] highlighted the challenges associated with establishing a universally accepted formulation for shear correction factors. Another significant class of refinement methods documented in the literature relies on the utilization of warping functions. Notable contributions in this area include the works of El Fatmi [20–22] and Ladevèze et al. [23, 24]. The three-dimensional free vibration of annular sector plates [25] and generalized super elliptical plates [26] with various boundary conditions by means of the Chebyshev–Ritz method is studied by Zhou et al. The analysis is based on the three-dimensional small strain linear elasticity theory. Rand [27] and Kim and White [28] employed a similar approach in the analysis of free vibration by introducing out-of-plane warping with no in-plane stretching terms. Asymptotic-type expansions in conjunction with variational methods have been proposed, particularly by Berdichevsky et al. [29]. They provide a commendable review of previous works on beam theory developments. Further valuable contributions in this field have been reported by Volovoi [30], Popescu and Hodges [31], and Yu et al. [32–34]. A novel quasi-3D hyperbolic theory is presented by Shahasvari et al. [35] for the free vibration analysis of functionally graded (FG) porous plates resting on elastic foundations by dividing transverse displacement into bending, shear, and thickness stretching parts. Another quasi three-dimensional (quasi-3D) shear deformation theories are presented by Akavci and Tanrikulu [36] for static and free vibration analysis of single-layer functionally graded (FG) plates using a new hyperbolic shape function. Additional relevant research can be found in the papers published by Kim and Wang [37] and Firouz-Abad et al. [38]. The generalized beam theory (GBT) is believed to have originated from the work of Schardt [39,40]. GBT improves classical theories by employing a piece-wise beam description for thin-walled sections. It has been widely utilized and further extended in various forms by Silvestre et al. [41–43], with a dynamic application being presented by Bebiano et al. [44]. Higher-order theories are typically developed by employing refined displacement fields for the beam cross-sections. Washizu [45] demonstrated how the use of a suitably chosen displacement field can lead to closed-form exact 3D solutions. Numerous other higher-order theories have also been proposed to incorporate non-classical effects [5,46–48].

The aforementioned works demonstrate a notable interest in exploring refined beam theories. In contrast to those publications, the Carrera unified formulation (CUF) is a hierarchical formulation that has been well established in the literature for over a decade [49–54]. The strength of CUF lies in its ability to facilitate the automatic development and compact formulation of any structural theory. This is achieved by expressing the 3D displacement field as a series expansion of the generalized unknowns, which are defined along the beam axis in the case of 1D models, through specific cross-sectional functions. A comprehensive discussion about CUF can be found in Carrera’s work [49]. Over the past years, the CUF has been applied to different problems by using Taylor expansion (TE) and Lagrange expansion (LE) as cross-sectional functions [55–59]. In the majority of these papers on 1D CUF, the finite element method (FEM) has been used to handle arbitrary geometries and loading conditions. However, the calculation accuracy of the FEM depends heavily on mesh refinement, which in turn restricts the calculation efficiency. In contrast to the 1D Carrera unified formulation (CUF) model solved using weak-form solutions such as FEM, analytical methods are not affected by meshing and can provide highly accurate dynamic analysis. Giunta et al. [60–62] presented a strong-form solution, known as the Navier-type solution, for the 1D CUF TE governing equation. They applied this solution to the free vibration analysis of composite beams [60] as well as the static, buckling, and free vibration analysis of sandwich beams [53,61,62]. The extension of the Navier-type closed-form solution to the 1D CUF LE for free vibration analysis of isotropic beams was done by Dan et al. [63]. However, the above analytic method is only limited to the special boundary conditions, e.g., simply supported. It is not easy to develop analytical solutions for the free vibration analysis of beams with solid and thin-walled cross-sections under arbitrary boundary conditions.

A powerful alternative tool has shown great potential for CUF theories through the application of the dynamic stiffness method (DSM) to carry out the free vibration analysis of solid and thin-walled structures in a much broader context by allowing for the cross-sectional deformation. Furthermore, the dynamic stiffness method (DSM) is applicable to beam structures with arbitrary boundary conditions. The DSM is often referred to as an exact method as it is based on the exact general solution of the governing differential equations [64–75]. This essentially means that, unlike the FEM and other approximate methods, model accuracy is not unduly compromised when a small number of elements are used in the analysis. For instance, one single structural element can be used in the DSM to compute any number of natural frequencies with any desired accuracy. It is worth mentioning that compared to other analytical methods, the DSM applies an efficient and robust algorithm, the Wittrick–Williams (WW) algorithm, which guarantees that no natural frequency is missed. DSM has been quite extensively developed for beam elements by Banerjee et al. [64,70,71] and Williams and Wittrick [68,71]. Liu and his co-authors have proposed DS theories for plate-like structures [76] subjected to general boundary conditions [77,78], stochastic boundary conditions [79], and beam built-up structures [80]. Recently, Pagani et al. [81–83] have established an exact dynamic stiffness formulation based on the use of Taylor type polynomials to define the displacement field above the beam cross-section. Each field consists of a direct extension to higher-order expansions of the Timoshenko beam theory. However, the use of Taylor-type expansions has some intrinsic limitations: (1) the introduced variables have a mathematical meaning (derivatives at the beam axes); (2) higher order terms cannot have a local meaning, they can have cross-section properties only; (3) the extension to large rotation formulation could experience difficulties. In addition, the J0 count in the WW algorithm is an important and difficult problem. J0 is the number of natural frequencies below the trial frequency when all the nodes of the structure are clamped. Previous researches discretized the structure into a finer dynamic stiffness mesh to ensure that J0 is equal to zero, which greatly reduces the computational efficiency and does not bring the merit of the DSM into full play.

To overcome these problems, this work combines the dynamic stiffness method (DSM) with beam theories that use Lagrange-type polynomial expansions to describe the displacement field of the cross-section. The use of these expansion functions allows for the representation of displacement variables only. This aspect is particularly advantageous because [55–57]: (1) each variable has a precise physical meaning (the problem unknowns are only translational displacements); (2) unknown variables can be put in fixed zones (sub-domains) of the cross-section area; (3) geometrical boundary conditions can be applied in sub-domains of the cross-section (and not only to the whole cross-section); (4) geometrical boundary conditions can also be applied along the beam-axis; (5) the extension to geometrically non-linear problems appears more suitable than in the case of Taylor-type higher-order theories. In particular, the component-wise (CW) approach based on Lagrange expansion is applied in which the solid part and thin-walled
part are considered as two independent components that can be assembled. In the CW approach, each component is modelled individually and simultaneously by using CUF beam elements (see [55]). Then, continuity conditions among the different components are automatically satisfied if Lagrange polynomials are used to approximate the cross-section kinematics. A recent successful application of the CW approach can be seen in [55]. In this work, the Principle of Virtual Displacements (PVD) is used to derive the differential governing equations and the associated natural boundary conditions for the LE model. By assuming harmonic oscillation, the equilibrium equations and the natural boundary conditions are formulated in the frequency domain by making extensive use of symbolic computation. The resulting system of ordinary differential equations of second order with constant coefficients is then solved in a closed analytical form. Subsequently, the frequency-dependent DS matrix of the system is derived by relating the amplitudes of the harmonically varying nodal generalized forces to those of the nodal generalized displacements. Finally, the mode count by the WW algorithm under support conditions is obtained in this study and extended WW algorithm is applied to the resulting DS matrix for free vibration analysis of solid beams, thin-walled structures and reinforced panel structures. This exact solution for free vibration analysis will be characterized by high efficiency in terms of computational costs and unprecedented accuracy.

The paper is organized as follows: a brief introduction of 1D CUF beam theory is given in Section 2. The governing equations of the LE model and analytical solution are derived using the principle of virtual work in Section 3. Section 4 presents the dynamic stiffness formulations and extended WW algorithm. Next, three examples taken from the literature are used to validate the proposed model in Section 5, from which the availability of CUF-DSM to provide 3D accuracy of solid beams, thin-walled structures and reinforced panels under different boundary conditions. Finally, some meaningful conclusions based on the above analysis are obtained in Section 6.

2. 1D CUF LE beam theory

2.1. Preliminaries

The adopted coordinate frame of the generic beam model is presented in Fig. 1. The beam has cross-section Ω and length L. The displacement vector is

\[ \mathbf{u}(x, y, z, t) = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T \]  

(1)

in which \( u_x, u_y, \) and \( u_z \) are the displacement components along \( x, y, \) and \( z \)-axes, respectively. The superscript “\( T \)” represents a transpose. The stress, \( \sigma \), and the strain, \( \epsilon \), components are grouped as follows

\[ \begin{aligned} \sigma &= \begin{bmatrix} \sigma_{yy} & \sigma_{xx} & \sigma_{xz} & \sigma_{yz} & \sigma_{xy} \end{bmatrix}^T, \\ \epsilon &= \begin{bmatrix} \epsilon_{yy} & \epsilon_{xx} & \epsilon_{xz} & \epsilon_{yz} & \epsilon_{xy} \end{bmatrix}^T \end{aligned} \]  

(2)

In the case of small displacements with respect to a characteristic dimension in the plane of \( \Omega \), the strain–displacement relations are

\[ \epsilon = \mathbf{D} \mathbf{u} \]  

(3)

where \( \mathbf{D} \) is the following linear differential operator matrix

\[ \mathbf{D} = \begin{bmatrix} 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \]  

(4)

According to Hooke’s law, the relationship between stress and strain is

\[ \sigma = \hat{\mathbf{C}} \epsilon \]  

(5)

In the case of isotropic material, the matrix \( \hat{\mathbf{C}} \) is

\[ \hat{\mathbf{C}} = \begin{bmatrix} C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\ C_{23} & C_{22} & C_{12} & 0 & 0 & 0 \\ C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{C}_{66} \end{bmatrix} \]  

(6)

Coefficients \( C_{ij} \) depend on Young’s modulus and Poisson’s ratio, which can be found in standard texts, see Reddy [84] or Tsai [85].

2.2. Unified formulation of beams

Within the framework of CUF, the 3D displacement field \( \mathbf{u}(x, y, z; t) \) can be expressed as an expansion of the generalized displacements through generic functions \( F_t \)

\[ \mathbf{u}(x, y, z; t) = F_t(x, y, z) \mathbf{u}_t(y, t), \quad t = 1, 2, \ldots, M \]  

(7)

where \( F_t \) are the functions of the coordinates \( x \) and \( z \) on the cross-section, \( \mathbf{u}_t \) is the generalized displacements vector and \( M \) stands for the number of terms in the expansion. According to the Einstein notation, the repeated subscript, \( t \), indicates summation. In this paper, Lagrange polynomials are used for \( F_t \) functions. In particular, 4-point (L4) bilinear, 9-point (L9) cubic and 16-point (L16) fourth-order polynomials are used. The order of the beam model is directly related to the choice of the \( F_t \) cross-sectional polynomial. Refined models of complex structures can also be implemented by considering cross-sectional assembly of those elements, such as in Fig. 2, where one L9 elements in actual geometry are shown. Moreover, the isoparametric formulation is exploited to deal with arbitrary shapes.

In the case of the L4 element, the interpolation functions are given by

\[ F_t = \frac{1}{4} \left(1 + r_t \right) \left(1 + s_t \right), \quad t = 1, 2, 3, 4 \]  

(8)

where \( r \) and \( s \) are the normalized coordinates that vary from \(-1\) to \(1\) and \( r_t \) and \( s_t \) are the actual coordinates of the four nodes.
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Then, the interpolation functions of 9-point (L9) cubic polynomial element are given by

\[
F_i = \frac{1}{2} r^2 + r r_i \left( s^2 + s s_i \right), \quad \tau = 1, 3, 5, 7
\]

\[
F_i = \frac{1}{2} s^2 + s s_i \left( 1 - r^2 \right) + \frac{1}{2} r \left( r^2 + r r_i \right) \left( 1 - s^2 \right), \quad \tau = 2, 4, 6, 8
\]

\[
F_i = \left( 1 - r^2 \right) \left( 1 - s^2 \right), \quad \tau = 9
\]

where \( r_i \) and \( s_i \) are the actual coordinates of the nine nodes.

Finally, the interpolation functions of 16-point (L16) fourth-order polynomial element are given by

\[
F_{m,n} = L_m(s)L_n(t), \quad m, n = 1, 2, 3, 4
\]

where

\[
L_1(r) = \frac{1}{5} (r-1)(1-9r^2), \quad L_2(r) = \frac{9}{5} (3r-1)(r^2-1)
\]

\[
L_3(r) = \frac{9}{5} (3r+1)(1-r^2), \quad L_4(r) = \frac{1}{5} (r+1)(9r^2-1)
\]

The cross-section displacement fields can be defined according to different elements and Eq. (7). For instance, the complete displacement field given by one single L4 element is

\[
u_s = F_1 u_{s1} + F_2 u_{s2} + F_3 u_{s3} + F_4 u_{s4}
\]

\[
u_s = F_1 u_{s1} + F_2 u_{s2} + F_3 u_{s3} + F_4 u_{s4}
\]

\[
u_s = F_1 u_{s1} + F_2 u_{s2} + F_3 u_{s3} + F_4 u_{s4}
\]

where \( u_{s1}, \ldots, u_{s4} \) are the unknown variables of the problem and represent the translational displacement components of each of the four points of the L4 element. The above displacement variables are the only unknowns, which do not lie on the beam element axis.

As already mentioned, the resulting LE can be used for the whole cross-section or can be introduced by dividing the cross-section into various sub-domains. The resulting approach is referred to as Component-Wise (CW) because Lagrange elements are used to model the displacement variables in each structural component at the cross-sectional level. Fig. 3 shows a possible CW model of the spar where each component is modelled via one LE element. Each LE element is then assembled above the cross-section to obtain the global stiffness matrix based on the 1D formulation. Since panels could not be reasonably modelled via a 1D formulation, 1D CW models can be refined by using several L-elements for one component. This methodology allows us to tune the capabilities of the model by (1) choosing which component requires a more detailed model; (2) setting the order of the structural model to be used.

3. Governing equations of the LE model and analytical solution

The principle of virtual displacements is used to derive the equations of motion

\[
\delta L_{\text{int}} = \int_V \delta e^T \sigma dV = -\delta L_{\text{ine}}
\]

where \( L_{\text{ine}} \) stands for the strain energy and \( L_{\text{ine}} \) is the work done by the inertial loadings. \( \delta \) stands for the usual virtual variation operator. The virtual variation of the strain energy is rewritten using Eqs. (3), (6) and (7). After integrations by part, Eq. (13) becomes

\[
\delta L_{\text{int}} = \int_V \delta u^T \mathbf{K}^{\tau s} \mathbf{u}, \quad dy + \left[ \delta u^T \mathbf{N}^{\tau s} \mathbf{u} \right]_{y=0}^{y=L}
\]

where \( \mathbf{K}^{\tau s} \) is the linear differential stiffness matrix and \( \mathbf{N}^{\tau s} \) is the matrix of natural boundary conditions. For the sake of brevity, these matrices are not given here but they can be found in [81–83]. The fundamental nuclei have the key property that their mathematical expressions remain unchanged regardless of the order of the beam theory or the choice of \( F \) functions.

The virtual variation of the inertial work is given by

\[
\delta L_{\text{ine}} = \int_L \delta u^T \rho F^s \mathbf{F} \delta u, \quad dy = \int_L \delta u^T \mathbf{M}^{\tau s} \mathbf{u}, \quad dy
\]

where \( \rho \) denotes the material density and double over dots stand for the second derivative with respect to time \( t \). \( \mathbf{M}^{\tau s} \) is the 3 \( \times \) 3 fundamental, diagonal nucleus of the mass matrix, whose components can be found in [40]. The explicit form of the governing equations is

\[
\delta u_{x, l} = -E_t^\tau \delta u_{x, l} + \left( E_t^{22} + E_t^{23} \right) u_{y, l} + \left( E_t^{33} - E_t^{23} \right) u_{z, l}
\]

\[
\delta u_{y, l} = -E_t^\tau \delta u_{y, l} + \left( E_t^{22} + E_t^{23} \right) u_{x, l} + \left( E_t^{33} + E_t^{23} \right) u_{z, l}
\]

\[
\delta u_{z, l} = -E_t^\tau \delta u_{z, l} + \left( E_t^{22} + E_t^{23} \right) u_{x, l} + \left( E_t^{33} + E_t^{23} \right) u_{y, l}
\]

The generic term \( E_t^\tau \) above is a cross-sectional moment parameter

\[
E_t^\tau = \int \mathbf{C}^\tau \mathbf{F}_t^s \mathbf{F} d\Omega
\]

The suffix after the comma denotes the derivatives. Moreover,

\[
\mathbf{P}_t = \left( P_{x, i} \quad P_{y, i} \quad P_{z, i} \right)^T
\]

Letting \( \mathbf{P}_t \) to be the vector of the generalized forces, the natural boundary conditions are

\[
\delta u_{x, l} = E_t^\tau \delta u_{x, l} + E_t^\tau \delta u_{y, l} + E_t^\tau \delta u_{z, l}
\]

\[
\delta u_{y, l} = E_t^\tau \delta u_{x, l} + E_t^\tau \delta u_{y, l} + E_t^\tau \delta u_{z, l}
\]

\[
\delta u_{z, l} = E_t^\tau \delta u_{x, l} + E_t^\tau \delta u_{y, l} + E_t^\tau \delta u_{z, l}
\]

For a fixed approximation order, Eqs. (16) and (19) have to be expanded using the indices \( \tau \) and \( s \) in order to obtain the governing differential equations and the natural boundary conditions of the desired model. In the case of harmonic motion, the solution of Eq. (16) is sought in the form

\[
u_s = e^{j\omega t} \mathbf{u}_s(t)
\]

where \( \mathbf{u}_s(t) \) is the amplitude function of the motion, \( \omega \) is an arbitrary circular or angular frequency, and \( t \) is the imaginary unit. Eq. (20) allows the formulation of the equilibrium equations and the natural boundary conditions in the frequency domain. Substituting Eq. (20) into Eq. (16), a set of three coupled ordinary differential equations is obtained which can be written in a matrix form as follows

\[
\delta \mathbf{U}_t = \mathbf{L}^\tau \delta \mathbf{U}_s = 0
\]

where

\[
\mathbf{U}_t = \left( U_{x, s} \quad U_{y, s} \quad U_{z, s} \quad U_{yx, s} \quad U_{yz, s} \quad U_{xz, s} \quad U_{yx, g} \quad U_{yz, g} \quad U_{xz, g} \right)^T
\]
The equations of motion can be obtained in the form of Eq. (23) as given below by expanding L^{11} as shown in Fig. 4. It reads as

\[ \mathbf{L} \dot{\mathbf{U}} = 0 \]  

(23)

In a similar way, the boundary conditions of Eq. (19) can be written in a matrix form as

\[ \delta \mathbf{U}_r = \mathbf{B}^{11} \dot{\mathbf{U}}_j \]  

(24)

where

\[ \dot{\mathbf{U}}_j = \begin{pmatrix} U_{x1} & U_{y1} & U_{z1} & U_{x2} & U_{y2} & U_{z2} \end{pmatrix}^T \]  

(25)

and \( \mathbf{B}^{11} \) is the 3 x 6 fundamental nucleus which contains the coefficients of the natural boundary conditions which are available from the corresponding literature \([82,83]\). For a given expansion order, the boundary conditions can be obtained in the form of Eq. (26) by expanding \( \mathbf{B}^{11} \) in the same way as \( L^{11} \) to finally give

\[ \mathbf{P} = \mathbf{B} \dot{\mathbf{U}} \]  

(26)

Eq. (23) is a system of ordinary differential equations (ODEs) of second order in \( y \) with constant coefficients. A change of variables is used to reduce the second order system of ODEs to a first order system.

\[ \mathbf{Z} = \begin{pmatrix} Z_1 & Z_2 & \ldots & Z_n \end{pmatrix}^T = \dot{\mathbf{U}} \]  

\[ \mathbf{Z} = \begin{pmatrix} U_{x1} & U_{x2} & U_{y1} & U_{y2} & U_{z1} & U_{z2} & \ldots & U_{xM} & U_{yM} & U_{zM} \end{pmatrix}^T \]  

(27)

where \( \dot{\mathbf{U}} \) is the expansion of \( \mathbf{U}_j \) for a given theory order, \( M \) is the number of expansion terms for the given DSM-LE theory, and \( n = 6 \times M \) is the dimension of the unknown vector as well as the number of differential equations. In \([82]\), an automatic algorithm to transform the \( L \) matrix of Eq. (23) into the matrix \( S \) of the following linear differential system was described

\[ \mathbf{Z}_1(y) = \mathbf{S} \mathbf{Z}(y) \]  

(28)

Once the differential problem is described in terms of Eq. (28), the solution can be written as follows

\[
\begin{bmatrix}
Z_1 \\
Z_2 \\
\vdots \\
Z_n
\end{bmatrix} =
\begin{bmatrix}
\delta_{11} & \delta_{12} & \cdots & \delta_{1n} & C_1 e^{\lambda_1 y} \\
\delta_{21} & \delta_{22} & \cdots & \delta_{2n} & C_2 e^{\lambda_2 y} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} & C_n e^{\lambda_n y}
\end{bmatrix}
\]  

(29)

where \( \delta_{ij} \) is the \( i \)th eigenvalue of the \( S \) matrix, \( \delta_{ij} \) is the \( j \)th element of the \( i \)th eigenvector of the \( S \) matrix and \( C_i \) are the integration constants which need to be determined by using the boundary conditions. The above equation can be written in matrix form as follows

\[ \mathbf{Z} = \mathbf{S} e^{\mathbf{C} y} \]  

(30)

Vector \( \mathbf{Z} \) does not only contain the displacements but also their first derivatives. If only the displacements are needed, according to Eq. (27), only the lines 1, 3, 5, ..., \( n-1 \) should be taken into account, giving a solution in the following form

\[
\begin{align*}
U_{x1}(y) &= C_1 \delta_{11} e^{\lambda_1 y} + C_2 \delta_{21} e^{\lambda_2 y} + \cdots + C_n \delta_{n1} e^{\lambda_n y} \\
U_{y1}(y) &= C_1 \delta_{12} e^{\lambda_1 y} + C_2 \delta_{22} e^{\lambda_2 y} + \cdots + C_n \delta_{n2} e^{\lambda_n y} \\
U_{z1}(y) &= C_1 \delta_{13} e^{\lambda_1 y} + C_2 \delta_{23} e^{\lambda_2 y} + \cdots + C_n \delta_{n3} e^{\lambda_n y} \\
&\vdots \\
U_{xM}(y) &= C_1 \delta_{1(M-1)} e^{\lambda_1 y} + C_2 \delta_{2(M-1)} e^{\lambda_2 y} + \cdots + C_n \delta_{n(M-1)} e^{\lambda_n y}
\end{align*}
\]  

(31)

Once the displacements are known, the boundary conditions are obtained by substituting the solution of Eq. (30) into the boundary conditions (Eq. (26)). In fact, it should be noted that \( \dot{\mathbf{U}} = \mathbf{Z} \) (Eq. (27)). It reads

\[ \mathbf{P} = \mathbf{B} \mathbf{S} \mathbf{C} e^{\mathbf{C} y} \]  

(32)

where \( \mathbf{A} = \mathbf{B} \mathbf{S} \). The boundary conditions can be written in explicit form as follows

\[
\begin{align*}
P_{x1}(y) &= C_1 A_{11} e^{\lambda_1 y} + C_2 A_{12} e^{\lambda_2 y} + \cdots + C_n A_{1n} e^{\lambda_n y} \\
P_{y1}(y) &= C_1 A_{21} e^{\lambda_1 y} + C_2 A_{22} e^{\lambda_2 y} + \cdots + C_n A_{2n} e^{\lambda_n y} \\
P_{z1}(y) &= C_1 A_{31} e^{\lambda_1 y} + C_2 A_{32} e^{\lambda_2 y} + \cdots + C_n A_{3n} e^{\lambda_n y} \\
&\vdots \\
P_{xM}(y) &= C_1 A_{M1} e^{\lambda_1 y} + C_2 A_{M2} e^{\lambda_2 y} + \cdots + C_n A_{Mn} e^{\lambda_n y}
\end{align*}
\]  

(33)
4. The dynamic stiffness formulation

4.1. Dynamic stiffness matrix

Once the closed form analytical solution of the differential equations of motion of the structural element in free vibration has been sought, a number of general boundary conditions which are usually the nodal displacements and forces — equal to twice the number of integration constants in algebraic form needs to be applied (see Fig. 5).

Starting from the displacements, the boundary conditions can be written as

\[ y = 0 : \]
\[ U_{11}(0) = \bar{U}_{11}, \quad U_{11}(L) = \bar{U}_{12} \]
\[ U_{12}(0) = \bar{U}_{12}, \quad U_{12}(L) = \bar{U}_{22} \]
\[ U_{1M}(0) = \bar{U}_{1M}, \quad U_{1M}(L) = \bar{U}_{2M} \]

\[ y = L : \]
\[ U_{11}(0) = \bar{U}_{11}, \quad U_{11}(L) = \bar{U}_{12} \]
\[ U_{12}(0) = \bar{U}_{12}, \quad U_{12}(L) = \bar{U}_{22} \]
\[ U_{1M}(0) = \bar{U}_{1M}, \quad U_{1M}(L) = \bar{U}_{2M} \]

(34)

By evaluating Eq. (31) at \( y = 0 \) and \( y = L \) and applying the boundary conditions of Eq. (34), the following matrix relation for the nodal displacements is obtained

\[
\begin{bmatrix}
\bar{U}_{11} \\
\bar{U}_{11} \\
\bar{U}_{12} \\
\vdots \\
\bar{U}_{1M}
\end{bmatrix} =
\begin{bmatrix}
\delta_{11} & \delta_{12} & \cdots & \delta_{1N} \\
\delta_{12} & \delta_{22} & \cdots & \delta_{2N} \\
\delta_{13} & \delta_{23} & \cdots & \delta_{3N} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{1(N-1)} & \delta_{2(N-1)} & \cdots & \delta_{N(N-1)}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\vdots \\
C_N
\end{bmatrix} =
\begin{bmatrix}
\bar{U}_{11} \\
\bar{U}_{12} \\
\bar{U}_{1M}
\end{bmatrix} =
\begin{bmatrix}
\delta_{11}e^{i\lambda L} & \delta_{12}e^{i\lambda L} & \cdots & \delta_{1N}e^{i\lambda L} \\
\delta_{12}e^{i\lambda L} & \delta_{22}e^{i\lambda L} & \cdots & \delta_{2N}e^{i\lambda L} \\
\delta_{13}e^{i\lambda L} & \delta_{23}e^{i\lambda L} & \cdots & \delta_{3N}e^{i\lambda L} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{1(N-1)}e^{i\lambda L} & \delta_{2(N-1)}e^{i\lambda L} & \cdots & \delta_{N(N-1)}e^{i\lambda L}
\end{bmatrix}
\begin{bmatrix}
\bar{U}_{11} \\
\bar{U}_{12} \\
\bar{U}_{1M}
\end{bmatrix} =
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\vdots \\
C_N
\end{bmatrix}

(35)

The above equation can be written in a more compact form as

\[
\bar{U} = AC
\]

(36)
Similarly, boundary conditions for generalized nodal forces are as follows:

\[ y = 0 \quad \text{and} \quad y = L : \]

\[ P_{x1}(0) = -\hat{P}_{x1} \]

\[ P_{x1}(L) = \hat{P}_{x1} \]

\[ P_{y1}(0) = -\hat{P}_{y1} \]

\[ P_{y1}(L) = \hat{P}_{y1} \]

\[ P_{z1}(0) = -\hat{P}_{z1} \]

\[ P_{z1}(L) = \hat{P}_{z1} \]

\[ \vdots \]

\[ P_{xM}(0) = -\hat{P}_{xM} \]

\[ P_{xM}(L) = \hat{P}_{xM} \]

By evaluating Eq. (33) at \( y = 0 \) and \( y = L \) and applying the BCs of Eq. (37), the following matrix relation for the nodal forces is obtained:

\[
\begin{bmatrix}
\hat{P}_{1x1} \\
\hat{P}_{1y1} \\
\vdots \\
\hat{P}_{1xM}
\end{bmatrix} =
\begin{bmatrix}
-A_{11} & -A_{12} & \cdots & -A_{1n} \\
-A_{21} & -A_{22} & \cdots & -A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-A_{n1} & -A_{n2} & \cdots & -A_{nn}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n
\end{bmatrix}
\]

\[ \hat{P}_{1x1} = \begin{bmatrix}
-A_{11} & -A_{12} & \cdots & -A_{1n} \\
-A_{21} & -A_{22} & \cdots & -A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-A_{n1} & -A_{n2} & \cdots & -A_{nn}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n
\end{bmatrix}
\]

The above equation can be written in a more compact form as

\[ \mathbf{P} = \mathbf{KU} \]

\[ \mathbf{P} = \mathbf{RA}^{-1} \]

where \( \mathbf{K} \) is the required DS matrix. The DS matrix given by Eq. (41) is the basic building block to compute the exact natural frequencies of a higher-order beam. The global DS matrix can be obtained by assembling elemental matrices as in the classical way of FEM [76]. In particular, it is possible to assemble elemental DS matrices to form the overall DS matrix of any complex structures consisting of beam elements (see Fig. 6).

The global DS matrix can be written as

\[ \mathbf{P}^G = \mathbf{K}^G \mathbf{U} \]

where \( \mathbf{K}^G \) is the square global DS matrix of the final structure. For the sake of simplicity, the subscript “G” is omitted hereafter. The boundary conditions can be applied by simply eliminating rows and columns of the dynamic stiffness matrix \( \mathbf{K}^G \) corresponding to the degrees of freedom which are zeros. Due to the presence of degrees of freedom at each interface, a multitude of boundary conditions can be applied at the required nodes. This means that once a node is constrained, all the cross-section points in correspondence to the node will be constrained. Different types of constraints can be applied in the CUF beam model, the most important implemented constrain types and the associated degrees of freedom that are applied are as follows:

- Free end (F): elimination no applied.
- Clamped end (C): elimination applied to \( U_{x1}, U_{y1}, U_{z1}, U_{x2}, U_{y2}, U_{z2}, \ldots, U_{x_n} \) at each end.
- Simply supported (S): elimination applied to \( U_{x1}, U_{y1}, U_{z1}, U_{x2}, U_{y2}, U_{z2}, \ldots, U_{x_n} \) at each end.

If necessary, it is also possible to constrain the given degrees of freedom of a certain point, as is usual in FE analysis.

4.2. The extended Wittrick–Williams algorithm and mode shapes computation

Once the dynamic stiffness matrix is developed, the Wittrick–Williams algorithm [86] can be applied to compute the natural frequencies \( \omega \) of structures. The following equation is the key equation of the Wittrick–Williams algorithm, which is used to calculate the mode count \( J \) when \( \omega \) is lower than the trial frequency \( \omega^* \)

\[ J = J_0 + s(\mathbf{K}(\omega^*)) \]

where \( \mathbf{K}(\omega^*) \) is the elemental dynamic stiffness matrix, \( s(\mathbf{K}(\omega^*)) \) is the number of negative diagonal elements after upper triangular transformation by using Gauss elimination of \( \mathbf{K}(\omega^*) \), and \( J_0 \) is the number of
natural frequencies between \( \omega = 0 \) and \( \omega = \omega^\ast \) when the nodal boundaries of the beam element are fully clamped. There is no doubt that \( J_0 \) plays an important role in the Wittrick–Williams algorithm. However, calculating \( J_0 \) is generally a difficult problem, and the traditional way is to refine the mesh to make sure \( J_0 = 0 \). Obviously, it will introduce unnecessary computational cost significantly.

In this study, the \( J_0 \) problem of the beam element is resolved by applying an indirect method, it improves the computational efficiency of the dynamic stiffness method. According to the Wittrick–Williams algorithm, the mode count \( J_i \) of the beam element with all simply supported (SS) when the half-wave number in the \( y \) direction is \( m \) can be given by Eq. (43), which can be recast as

\[
J_0 = \sum_{i=1}^{m} J_{0m}
\]

\[
J_{0m} = J_{0m} - s \left( K'(\omega^\ast) \right)
\]

where \( J_{0m} \) is \( J_0 \) when the half-wave number in the \( y \) direction is \( m \) and \( K'(\omega^\ast) \) is the dynamic stiffness matrix \( K(\omega^\ast) \) for a beam element with all simply supported when \( m \) is a certain value. The detailed solution process for \( J_{0m} \) is given as follows: The first step is to solve the natural frequencies \( \omega^\ast \) of the beam element under certain \( m \) for SS boundary conditions. Consider the BC of the beam element is SS, the displacement fields are assumed as a sum of harmonic functions

\[
\begin{align*}
\delta u_x(y,t) &= U_x e^{i\omega t} \sin(\pi y/L) \\
\delta u_y(y,t) &= U_y e^{i\omega t} \cos(\pi y/L) \\
\delta u_z(y,t) &= U_z e^{i\omega t} \sin(\pi y/L)
\end{align*}
\]

where \( \omega \)

\[
\omega = \frac{m\pi}{L}
\]

By substituting Eq. (46) into the Eq. (16), it holds

\[
\begin{align*}
\delta u_{xs} &= \left( a^2 E_{11} + E_{22} + E_{44} + (-\omega^2 E_{15}) \right) u_{sx} \\
&\quad - a \left( E_{23} E_{66} - E_{66} \right) U_{ss} + \left( E_{44} + E_{15}^2 \right) U_{sz} = 0 \\
\delta u_{ys} &= a \left( E_{66} - E_{44} \right) U_{sx} + \left( a^2 E_{15} + E_{66} + E_{55} \right) U_{sy} \\
&\quad - a^2 E_{15} U_{ys} + a \left( E_{55} - E_{15} \right) U_{sz} = 0 \\
\delta u_{zs} &= \left( E_{44} + E_{12} \right) U_{sx} \quad + a \left( E_{13} - E_{55} \right) U_{ys} + \left( a^2 E_{15} + E_{66} \right) U_{sz} = 0
\end{align*}
\]

The above equations can be converted into the algebraic eigensystem as

\[
(K^S - \omega^2 M^S)U = 0
\]

where \( K^S \) and \( M^S \) are the fundamental nuclei of the algebraic stiffness and mass matrices, respectively. The components of the linear stiffness matrix \( K^S \) are given in Eq. (50) (see Box I).

The components of the linear mass matrix \( M^S \) are

\[
M^S = \begin{bmatrix}
E_{11} & 0 & 0 \\
0 & E_{13} & 0 \\
0 & 0 & E_{15}
\end{bmatrix}
\]

Once the natural frequencies \( \omega^S \) of the beam element under certain \( m \) for SS boundary conditions are computed, \( J_{0m} \) is equal to the number of these frequencies \( \omega^S \) that are lower than the trial frequency \( \omega^\ast \). Finally, through Eqs. (43), (44) and (45), the natural frequency of the beam under different boundary conditions can be calculated accurately and efficiently. After computing the natural frequency and
evaluating the global dynamic stiffness (DS) matrix, the corresponding nodal generalized displacements can be determined by solving the associated homogeneous system of Eq. (42). By utilizing the nodal generalized displacements \( \mathbf{U} \), the integration constants C of the element can be calculated using Eq. (36). Subsequently, employing Eq. (31), the unknown generalized displacements can be obtained as a function of \( y \). Finally, by employing Eqs. (1) and (20), the complete displacement field can be generated as a function of \( x, y, z \), and time \( t \) (if an animated plot is required). The plot of the desired mode and element can be visualized on a fictitious 3D mesh.

5. Numerical results

The accuracy and computational efficiency of the present approach are demonstrated by carrying out the free vibration analysis of both solid and thin-walled structures in this section. In Section 5.1, free vibration of beams with rectangular cross-section are addressed so as to make an easy and straightforward comparisons with classical beam theories, CUF(TE)-DSM, CUF(LE)-Navier theory and reference FE results. In Section 5.2, a thin-walled C-shaped cross-section beam is considered. Free vibration analysis is carried out for different BCs and the present CUF(LE)-DSM models are also compared with reference solutions from the literature together with the results obtained from the FE commercial software ABAQUAS.

5.1. Solid beam structures

A beam with a solid rectangular cross-section are considered for preliminary assessments. The LE cross-sectional discretizations with Lagrange elements are depicted in Fig. 7 for the problem under consideration. Unless differently specified, “\( b \) = 0.2 m” is used. The isotropic material data are: Young modulus, \( E = 75 \) GPa, Poisson ratio, \( \nu = 0.33 \), material density, \( \rho = 2700 \) kg/m\(^3\).

Table 1 shows the first seven non-dimensional flexural frequencies \( \omega^* = (\omega L^2/b)\sqrt{\rho/E} \) for the simply supported (SS) square beam with \( L/b = 10 \). The results from the present LE models are compared to those from classical theories (EBBM, TBM, CUF(TE)-DSM, CUF(LE)-Navier theory and reference FE results from Refs. [63,82]. Various LE models are considered in the table as shown in Fig. 7. The comparison of the results in Table 1 shows the correctness of the present strong form LE beam. Even the most simple one, 1L4, demonstrates its convergence with respect to EBBM. The results obtained by the present method are in good agreement with the Navier solution which also uses LE model. Attention should be paid to 1L16, which converges slightly more precise results, at least in the range of the lower frequencies. The results of the 1L16 model show the higher-interpolation, fourth-order capabilities, owning the best accuracy, which is particularly evident in the higher frequencies range.

Fig. 8 shows the first four flexural modes of the beam with SS boundary conditions obtained from the CUF-DSM analysis when using a 1L9 LE model. It should be emphasized that DS results are mesh independent and the mesh used in Fig. 8 is merely a plotting grid for convenience. Those figures clearly demonstrate the 3D capabilities of the present CUF-DSM beam formulation.

One of the most important features of the DSM is that it provides exact solutions for any kind of boundary conditions. Moreover, LE
Fig. 7. LE modelling of the square cross-section beam. (a) 1L4. (b) 1 × 2L4. (c) 2 × 1L4. (d) 2 × 2L4. (e) 1L9. (f) 1L16.

Fig. 8. First (a), second (b), third (c) and fourth (d) flexural modes for a SS square beam ($L = b \times 10$), L9.

Theories are able to take into account several non-classical effects such as warping, in-plane deformations, shear effects and flexural-torsion couplings. Table 2 shows the first two flexural modes and the first two torsional modes for a clamped–free (CF) square beam for $L/b = 10$. The exact solutions for classical, linear and higher-order beam theories are also shown and they were computed using the DSM [82]. The CUF(TE)-DSM and reference FE results were computed from Ref. [82]. It can be seen that the present simple one (1L4) LE model is able to characterize the flexural behaviour of solid cross-section beams. A higher-interpolation (1L16) LE model is necessary to correctly detect torsional frequencies.

Fig. 9 shows some representative modal shapes for the 1L9 LE model of the CF beam. Those figures also demonstrate the 3D capabilities of the present CUF-DSM beam formulation for both flexural and torsional modes with any kind of boundary conditions.

5.2. Thin-walled beam structures

Free vibration analyses of a thin-walled C-shaped beam were carried out next for the assessment of the present beam model. The geometry of the cross-section is shown in Fig. 10(a). The sides of the cross-section are $a = 0.2$ m and $b = a$. The thicknesses are $t = a/10$, and the length-to-side ratio $L/a = 10$. The material data are: Young modulus, $E = 198$ GPa; Poisson ratio, $\nu = 0.3$; Material density $\rho = 7850$ kg/m$^3$. Various order LE models are considered in the following analysis and some cross-sectional discretizations are shown in Fig. 10(b)–(f) for illustrative purposes.

Table 3 shows the first eight natural frequencies (Hz) of the SS C-shaped cross-section beam. The results from the present LE models are compared to those from CUF(LE)-Navier theory and reference FE results from Ref. [63]. Various LE models are considered in the table as shown in Fig. 10. L4 models from 5 to 14 elements, and L9 models from 5 to 14 elements prove the accuracy of the proposed solution by comparing with the CUF(LE)-Navier solution. The results show that L9 models are affected by errors that are lower than 3% with respect to the 3D FEM solutions. On the other hand, considerable errors are produced by lower-order beam models in the higher frequencies range. The reason lower order L4 models do not give good results is that models from 5L4 to 14L4 do not have enough DOFs to characterize the shell-like modes. This aspect is clarified from Fig. 11, which shows the modes of vibrations by the 5L9 model.
Fig. 9. First flexural (a), second flexural (b), first torsional (c) and second torsional (d) modes for a CF square beam \((L = 6 \times 10)\), L9.

Fig. 10. C-shaped cross-section and LE discretizations. (a) C-shaped cross-section. (b) 5L4. (c) 14L4. (d) 5L9. (e) 8L9. (f) 14L9.

Fig. 11 shows the first nine modes of the beam with SS boundary conditions obtained from the CUF-DSM analysis when using a 5L9 LE model. The first three modes (see Fig. 11(a)–(c)) are flexural mode, while all the later modes (Fig. 11(d)–(i)) show the coupled modes. The ability of 1D DSM-LE models in dealing with 2D shell-like analyses is documented in the present works. It is worth noting that arbitrarily complex modes can be obtained by using only one element in the length direction by using the DSM, because the DSM provides the exact solution of the differential equations of the motion once the structural model has been formulated. In addition, each variable has a precise physical meaning by using Lagrange-type polynomials.

In order to prove that DSM-LE can also provide accurate solutions for thin-walled beam structures with different boundary conditions, Table 4 shows first eight natural frequencies (Hz) related of the CF C-shaped cross-section beam for \(L/a = 10\). Various LE models are considered in the table as shown in Fig. 10. FEM results from various ABAQUS models are also shown in Table 4. The ABQ C3D20 model with 125 706 DOFs (finer) and 62 854 DOFs (coarse) is adopted. From rows 3 to 7, the 14L9 model always shows the best accuracy among five different LE models because it has enough DOFs. In the first five natural frequencies, the percentage differences between the L4 models and the ABQ C3D20 model are small. Starting from the 6th frequency, the L4 model cannot produce reasonable results anymore. Especially for the 8th frequency, the percentage difference of the 5L4 and 14L4 model from the ABQ C3D20 model are 84.81% and 48.41%, respectively. The reason the L4 model cannot give good results is that L4 models do not have enough DOFs to characterize the shell-like modes. This aspect is also clarified from Fig. 12, which shows the modes of vibrations by the 5L9 model.

Fig. 12 shows the first eight modes of the beam with CF boundary conditions obtained from the CUF-DSM analysis when using a 5L9 LE model. The first five modes (see Fig. 12(a)–(e)) are uncoupled mode,
Table 3

<table>
<thead>
<tr>
<th>Model</th>
<th>Natural frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>FEM (63)</td>
<td></td>
</tr>
<tr>
<td>3D FEM coarse</td>
<td>69.019</td>
</tr>
<tr>
<td>3D FEM finer</td>
<td>68.894</td>
</tr>
<tr>
<td>CUF (LE)-Navier (63)</td>
<td></td>
</tr>
<tr>
<td>5 × L4</td>
<td>71.399</td>
</tr>
<tr>
<td>14 × L4</td>
<td>69.700</td>
</tr>
<tr>
<td>5 × L9</td>
<td>69.360</td>
</tr>
<tr>
<td>8 × L9</td>
<td>69.158</td>
</tr>
<tr>
<td>14 × L9</td>
<td>68.995</td>
</tr>
<tr>
<td>Current theory</td>
<td></td>
</tr>
<tr>
<td>5 × L4</td>
<td>71.399</td>
</tr>
<tr>
<td>14 × L4</td>
<td>69.700</td>
</tr>
<tr>
<td>5 × L9</td>
<td>69.360</td>
</tr>
<tr>
<td>8 × L9</td>
<td>69.158</td>
</tr>
<tr>
<td>14 × L9</td>
<td>68.995</td>
</tr>
</tbody>
</table>

Fig. 11. Uncoupled (a–c) and coupled (d–f) modal shapes for a SS C-shaped cross-section beam (\(L = a \times 10\), SL9.

Table 4

<table>
<thead>
<tr>
<th>Model</th>
<th>DOFs</th>
<th>Natural frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>FEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3D FEM coarse</td>
<td>62,854</td>
<td>32.233</td>
</tr>
<tr>
<td>3D FEM finer</td>
<td>125,706</td>
<td>32.089</td>
</tr>
<tr>
<td>Current theory</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 × L4</td>
<td>60</td>
<td>33,253</td>
</tr>
<tr>
<td>14 × L4</td>
<td>150</td>
<td>32,600</td>
</tr>
<tr>
<td>5 × L9</td>
<td>165</td>
<td>32,443</td>
</tr>
<tr>
<td>8 × L9</td>
<td>255</td>
<td>32,278</td>
</tr>
<tr>
<td>14 × L9</td>
<td>435</td>
<td>32,182</td>
</tr>
</tbody>
</table>

In order to further reflect the computational efficiency of this method, Table 5 shows first seven natural frequencies (Hz) related of the CF C-shaped cross-section beam for \(L/a = 5\). Various LE models are considered in the table as shown in Fig. 10. FEM results from ABQ

while all the later modes (Fig. 12(f)–(h)) show the coupled modes. The ability of 1D DSM-LE models in dealing with 2D shell-like analyses for thin-walled beam structures with different boundary conditions is documented in the present works.
Table 5

<table>
<thead>
<tr>
<th>Model</th>
<th>DOFs</th>
<th>Natural frequencies</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>FEM</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3D FEM coarse</td>
<td>31292</td>
<td>93.452</td>
<td>170.981</td>
</tr>
<tr>
<td>3D FEM finer</td>
<td>62984</td>
<td>93.194</td>
<td>170.592</td>
</tr>
<tr>
<td>Current theory</td>
<td></td>
<td>5 × L4</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14 × L4</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5 × L9</td>
<td>165</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8 × L9</td>
<td>255</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14 × L9</td>
<td>435</td>
</tr>
</tbody>
</table>

C3D20 models with 62854 DOFs (finer) and 31292 DOFs (coarse) are also shown in Table 5. The results illustrate that the DSM-LE can achieve comparable accuracy to a FEM model while requiring a significantly shorter calculation time, typically around 1 to 2 times faster. This is even more remarkable considering that the FEM model uses an optimized commercial solver while the DSM-LE runs as a research code in Matlab. This is due to the \( J_0 \) count solution technique presented in this paper, which greatly improves the computational efficiency, and the merit of the DSM is brought into full play.

5.3. Stiffened panel structures

Free vibration analyses of a stiffened panel were carried out next for the assessment of the present approach. A stiffened panel consisting of the stringer component and panel component is a common research object which is shown as Fig. 13. In the CW approach, each component is modelled individually and simultaneously by using CUF beam elements and the geometry of the cross-section is shown in Fig. 14(a). The stiffened panel structures are modelled as one-way slabs. The structures’ cross-section on the \( xz \)-plane are connected by two components of different cross-sections (Fig. 14), which adopt the type of common nodes. The boundary conditions of stiffened panel structures are applied at the ending nodes (edges \( y = 0 \) and \( y = b \)) and the \( x = 0 \) and \( x = a \) edges are inherently free in the case of beam models (see Fig. 1). The sides of the cross-section are \( a_1 = 0.096 \) m, \( a_2 = 0.032 \) m, \( b_1 = 0.1 \) m and \( b_2 = 0.2 \) m. The length of total panel is 1.5 m. The relevant material properties are shown in Table 6. Various order LE models are considered in the following analysis and some cross-sectional discretizations are shown in Fig. 14(b)–(f) for illustrative purposes.

Table 7 shows the first six natural frequencies (Hz) of the stiffened panel for different boundary conditions (BCs), namely, simply supported (SS) and clamped-free (CF) BCs. The solutions for various LE models considering in Fig. 14 are obtained using the CUF(LE)-DSM. The results are compared to FEM solutions from ABAQUS models which are referred to as ABQ C3D20 model with 91 423 DOFs (coarse) and 182 845 DOFs (finer). In the analysis, the 10L9 model always shows the best accuracy among five different LE models because it has enough DOFs. In the first one, three, four and six natural frequencies (flexural modes), the percentage differences between the L4 models and the ABQ C3D20 model are small. In the first two and five natural frequencies (shell-like modes), the L4 model and 7L9 model cannot produce reasonable results anymore. The reason the is that L4 models
Fig. 13. Component-wise approach for a stiffened panel structure.

Fig. 14. Stiffened panel cross-section and LE discretizations. (a) Stiffened panel cross-section. (b) 8L4. (c) 7L9. (d) 14L4. (e) 8L9. (f) 10L9.

and 7L9 model do not have enough DOFs to characterize the shell-like modes under different boundary conditions.

In addition, the first six modal shapes for a SS stiffened panel structure attained from 8L9 models are depicted in Fig. 15. The first one, three, four and six modes are flexural modes, respectively. The two and five modes are shell-like modes. It has been demonstrated from Table 7 and Fig. 15 that 1D higher-order models based on LE model are necessary to detect shell-like modes of combined structures such as reinforced panel.

5.4. Reinforced panels with four stiffeners

This section extends the use of the present approach to a reinforced panel structures. As far as the extension of the present methodology to plate-like structures is concerned, it is clear that the edges $x = 0$ and $x = a$ have to necessarily be free, since boundary conditions can only be applied at ending nodes (edges $y = 0$ and $y = b$) in the case of beam models (see Fig. 1). However, this limitation can be acceptable for most practical problems. The reinforced panel structures are modelled as one-way slabs which are connected by two different cross-sections

<table>
<thead>
<tr>
<th>Table 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>First six natural frequencies (Hz) of the stiffened panel for different boundary conditions.</td>
</tr>
<tr>
<td>BCS</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>SS</td>
</tr>
<tr>
<td>3D FEM finer 182845</td>
</tr>
<tr>
<td>8 × L4</td>
</tr>
<tr>
<td>14 × L4</td>
</tr>
<tr>
<td>7 × L9</td>
</tr>
<tr>
<td>8 × L9</td>
</tr>
<tr>
<td>10 × L9</td>
</tr>
<tr>
<td>CF</td>
</tr>
<tr>
<td>3D FEM finer 182845</td>
</tr>
<tr>
<td>8 × L4</td>
</tr>
<tr>
<td>14 × L4</td>
</tr>
<tr>
<td>7 × L9</td>
</tr>
<tr>
<td>8 × L9</td>
</tr>
<tr>
<td>10 × L9</td>
</tr>
</tbody>
</table>
Fig. 15. Flexural (a,c,d,f) and shell-like (b,e) modal shapes for a SS stiffened panel structure ($L = a_1 \times 15$), 8L9.

Fig. 16. Reinforced panels with four stiffeners.

Fig. 17. The geometry of the reinforced panels with four stiffeners.

(Fig. 17(b) and (c)) on the y-direction, which adopt the type of common nodes. For purpose of description, in this work the same notation as used in [81] has been adopted. Therefore, the symbolism C-F-F-F identifies a plate with the edges $y = 0, x = a, y = b$, and $x = 0$ having clamped, free, free, free boundary conditions, respectively (see Fig. 1). Similarly S-F-S-F denotes a plate with two opposite simply-supported edges and two opposite free edges. The dynamic analysis of a reinforced panel with four stiffeners as shown in Fig. 16 is presented and the geometry of the structure is shown in Fig. 17. The sides of the cross-section are $a = 0.5$ m, $b = 0.5$ m, $h_1 = h_2 = 0.2$ m and $t = 0.2$ m. The material data are: Young modulus, $E = 198$ GPa; Poisson ratio, $\nu = 0.3$; Material density $\rho = 7850$ kg/m$^3$. Various order LE models are considered in the following analysis and some cross-sectional discretizations are shown in Fig. 18(a)–(c) for illustrative purposes.

Table 8 shows the first seven natural frequencies (Hz) of the reinforced panel with four stiffeners for different boundary conditions (BCs), namely, S-F-S-F and C-F-F-F BCs. The solutions for various LE
Table 8
The first seven natural frequencies (Hz) of the reinforced panel with four stiffeners for different boundary conditions (BCs), namely, simply supported (SS) and clamped–free (CF) BCs.

<table>
<thead>
<tr>
<th>BCs</th>
<th>Model</th>
<th>DOFs</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
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<tr>
<td>SS</td>
<td>3D FEM coarse</td>
<td>10996</td>
<td>177.86</td>
<td>231.98</td>
<td>541.18</td>
<td>578.89</td>
<td>585.71</td>
<td>603.43</td>
<td>754.92</td>
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<tr>
<td></td>
<td>3D FEM finer</td>
<td>38845</td>
<td>176.97</td>
<td>228.83</td>
<td>539.40</td>
<td>573.71</td>
<td>580.14</td>
<td>593.68</td>
<td>751.05</td>
</tr>
<tr>
<td></td>
<td>7 × L9 &amp; 10 × L9</td>
<td>555</td>
<td>178.75</td>
<td>233.37</td>
<td>542.03</td>
<td>583.69</td>
<td>587.88</td>
<td>606.15</td>
<td>755.76</td>
</tr>
<tr>
<td></td>
<td>10 × L9 &amp; 16 × L9</td>
<td>825</td>
<td>178.02</td>
<td>231.62</td>
<td>540.92</td>
<td>578.16</td>
<td>585.13</td>
<td>602.27</td>
<td>753.23</td>
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<tr>
<td></td>
<td>13 × L9 &amp; 22 × L9</td>
<td>1095</td>
<td>177.35</td>
<td>231.02</td>
<td>540.13</td>
<td>576.79</td>
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<td>752.49</td>
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<tr>
<td>CF</td>
<td>3D FEM coarse</td>
<td>10996</td>
<td>70.655</td>
<td>135.89</td>
<td>252.97</td>
<td>357.35</td>
<td>443.92</td>
<td>459.93</td>
<td>630.37</td>
</tr>
<tr>
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<td>3D FEM finer</td>
<td>38845</td>
<td>70.031</td>
<td>132.74</td>
<td>252.01</td>
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<td>455.05</td>
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<td>70.705</td>
<td>135.66</td>
<td>253.24</td>
<td>357.93</td>
<td>444.61</td>
<td>463.92</td>
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<tr>
<td></td>
<td>10 × L9 &amp; 16 × L9</td>
<td>825</td>
<td>70.505</td>
<td>134.41</td>
<td>252.56</td>
<td>356.45</td>
<td>440.85</td>
<td>458.94</td>
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<td>70.309</td>
<td>134.03</td>
<td>252.39</td>
<td>355.88</td>
<td>439.78</td>
<td>457.98</td>
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</table>

Fig. 18. Reinforced panels cross-section and LE discretizations. (a) 7L9 & 10L9. (b) 10L9 & 16L9. (c) 13L9 & 22L9.

Fig. 19. The first four mode shapes for a SS reinforced panels with four stiffeners evaluated with FEM models.
models considering in Fig. 18 are obtained using the CUF(LE)-DSM. The results are compared to FEM solutions from ABAQUS models which are referred to as ABQ C3D20 model with 10 996 DOFs (coarse) and 38 845 DOFs (finer). The comparison between the results by FEM and those from the present models demonstrate the efficiency of the CUF(LE)-DSM models, in fact, a higher accuracy is reached with a reduction in the computational costs. In the analysis, the $13 \times 19$ and $22 \times 19$ model always shows the best accuracy among five different LE models because it has enough DOFs.

Fig. 19 shows the first four mode shapes for a S-F-S-F reinforced panels with four stiffeners evaluated with FEM models. 3D FEM model is used as a reference solution which is made using only 3D elements for all the components of the plate and stringers. The results from the present approach are comparable with those from the 3D FEM model in terms of accuracy, but can be achieved using 13% of the degrees of freedom.

6. Conclusions

The exact CUF-DS models has been developed for the free vibration analyses of solid beams, thin-walled structures and reinforced panels subjected to various boundary conditions. By exploiting the Carrera unified formulation (CUF) and Lagrange polynomials to discretize the beam cross-sectional kinematics, refined models with only displacement variables have been developed. The resulting DS matrix is applied using the extended Wittrick–Williams algorithm to compute the natural frequencies and mode shapes of some solid and thin-walled structures. Using these approaches, higher order models that are able to deal with shear deformation and higher-order effects, such as warping, can be captured straightforwardly with the help of CUF. Comparison studies are provided to validate the accuracy and cost-effectiveness of the proposed method. The following considerations arise from the comparison of the present approach with results available in the literature and 3D FEM solutions from commercial software:

• A higher-order exact DS formulations have been developed by first using Lagrange polynomials to define the displacement field above the cross-section of the beam. The Lagrange-based formulation offers enhanced capabilities compared to Taylor-based CUF modelling. The higher the Lagrange polynomials order, the better the accuracy.

• The resulting LE-DSM based on Component-Wise (CW) can be used for the whole cross-section or can be introduced by dividing the cross-section into various sub-domains, because Lagrange elements can be used to model the displacement variables in each structural component at the cross-sectional level. This characteristic allows us to separately model, for instance, stringers and panels in structures design.

• Analytical expressions of the $J_0$ count have been developed for the Wittrick–Williams (WW) algorithm. With the $J_0$ problem resolved, there is no need to split a large dynamic stiffness element into smaller ones unnecessarily as the majority of existing works did. Therefore, very few DOFs are required for beam structures, which has made the DSM to be highly efficient.

• Lower-order models are effective for symmetrical cross-sections like squares, but less so for asymmetrical structures such as C-shaped beams. Higher-order CUF LE models with sufficient degrees of freedom (DOFs) accurately capture shell-like modes in reinforced panels. Compared to the 3D FEM model, our approach achieves similar precision for reinforced panel analysis but requires only 13% of the DOFs, demonstrating efficiency and accuracy.

The results agree with those obtained using ABAQUS FE models and with those from the literature. The present analytical CUF(LE)-DSM formulation clearly demonstrates its efficiency over 3D FEM solutions and the capabilities of capturing higher-order refined effects. The investigation provides optimism for future studies on the dynamic analysis of composite structures.

CRediT authorship contribution statement

Xiao Liu: Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis. Alfonso Pagani: Writing – review & editing, Supervision, Formal analysis, Data curation, Conceptualization. Erasmo Carrera: Methodology, Conceptualization. Xiang Liu: Writing – review & editing, Supervision, Project administration, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Data will be made available on request.

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